Problem 6. Let \( \{a_n\} \) be an sequence and suppose \( \{a_{2n}\} \) and \( \{a_{2n-1}\} \) both converge to the same number, \( L \). Prove that \( \{a_n\} \) converges to \( L \).

Proof: Let \( \epsilon > 0 \). There exists \( K \) such that for all \( k \geq K \),

\[
|a_{2k} - L| < \epsilon \quad \text{and} \quad |a_{2k-1} - L| < \epsilon
\]

Let \( N \) be such that \( N \geq 2K \) \((N/2 \geq K)\), and let \( n \geq N \). If \( n \) is even, then \( n = 2k \) for some \( k \in \mathbb{N} \). Since \( n \geq N \), it follows that \( 2k \geq N \) and so \( k \geq N/2 \geq K \). But \( k \geq K \) implies

\[
|a_n - L| = |a_{2k} - L| < \epsilon
\]

Similarly, if \( n \) is odd, then \( n = 2k - 1 \) for some \( k \in \mathbb{N} \). But \( n \geq N \) implies \( 2k - 1 \geq N \), or \( k \geq (N + 1)/2 \geq N/2 \geq K \). Thus,

\[
|a_n - L| = |a_{2k-1} - L| < \epsilon
\]

Thus, in all cases, \( |a_n - L| < \epsilon \), for all \( n \geq N \). \( \square \)

Alternate Solution: Suppose \( \{a_n\} \) does not converge to \( L \), then from Problem 14, there exists an \( \epsilon > 0 \) and a subsequence \( \{a_{p_n}\} \) such that \( |a_{p_n} - L| \geq \epsilon \), for all \( n \in \mathbb{N} \). Now, it must be true that either infinitely many of the terms of \( \{p_n\} \) are even, or infinitely many of the terms of \( \{p_n\} \) are odd. WLOG, assume the former is true, and let \( \{q_n\} \) be the subsequence of \( \{p_n\} \) consisting of the even numbers in the sequence \( \{p_n\} \). Then \( \{a_{q_n}\} \) is a subsequence of \( \{a_{p_n}\} \) and so

\[
|a_{q_n} - L| \geq \epsilon \quad \forall n \in \mathbb{N}
\]

But \( \{a_{q_n}\} \) is also a subsequence of \( \{a_{2n}\} \), hence, by Theorem 2.17(a), it follows \( \{a_{q_n}\} \) converges to \( L \), this contradicts (3). \( \square \)

Problem 9. Let \( \{a_n\} \) be an unbounded sequence. Then there exists a subsequence \( \{a_{p_n}\} \) of \( \{a_n\} \) such that \( \{1/a_{p_n}\} \) converges to zero.

Proof: We construct the subsequence by induction. There exists an natural number \( p_1 \) such that \( |a_{p_1}| > 1 \). Suppose \( p_1 < p_2 < \cdots < p_n \) have been chosen so that \( |a_{p_k}| > k \), for \( k = 1, 2, \ldots, n \). Since \( n + 1 \) is not an upper bound for the sequence \( \{a_n\} \), there exists \( p_{n+1} \in \mathbb{N} \) such that \( |a_{p_{n+1}}| > n + 1 \). We can choose \( p_{n+1} > p_n \), because if such an index did not exist, it would follow that the sequence \( \{a_n\} \) is bounded. (Verify?) This completes the induction and the construction.

We have constructed a subsequence \( \{a_{p_n}\} \) of \( \{a_n\} \) with the property that \( |a_{p_n}| > n \), for all \( n \in \mathbb{N} \). It follows that \( |1/a_{p_n}| < 1/n \), or \( -1/n < 1/a_{p_n} < 1/n \). From the Squeeze Theorem, we deduce \( \{1/a_{p_n}\} \) converges to zero. \( \square \)
Problem 14. If \( \{x_n\} \) does not converge to \( L \), there exists an \( \epsilon > 0 \) and a subsequence \( \{x_{p_n}\} \) such that \( |x_{p_n} - L| > \epsilon \), for all \( n \in \mathbb{N} \).

Proof: Since \( \{x_n\} \) does not converge to \( L \), \( \exists \epsilon > 0 \) such that \( \forall N \in \mathbb{N}, \exists n \geq N \) such that \( |x_n - L| \geq \epsilon \). (This is the denial of the definition of convergence of \( \{x_n\} \) to \( L \).

We construct the subsequence by induction. Take \( N = 1 \), then there is \( p_1 \geq 1 \) such that \( |x_{p_1} - L| \geq \epsilon \). Suppose \( p_1 < p_2 < \cdots < p_n \) have been chosen so that \( |x_{p_k} - L| \geq \epsilon \), for \( 1 \leq k \leq n \). Choose \( N = p_n + 1 \), then there exists a natural number \( p_{n+1} \geq p_n + 1 \) such that \( |x_{p_{n+1}} - L| \geq \epsilon \). This completes the induction and the construction.

We have constructed a subsequence \( \{x_{p_n}\} \) of \( \{x_n\} \) such that \( |x_{p_n} - L| \geq \epsilon \), for all \( n \in \mathbb{N} \).

The only other issue is the strict inequality as stated in the problem. This is trivial. \( \Box \)

Problem 17(a)(b). We need compute limit the inferior and limit superior of two sequence.

(a) Let \( x_n = (n/3) - \lfloor n/3 \rfloor \).

Solution: If you calculate the first few terms of this sequence you get: \( 1/3, 2/3, 0, 1/3, 2/3, 0, \ldots \). This sequence takes on only three distinct values. Each of these three values—\( 0, 1/3, \) and \( 2/3 \)—are subsequential limit points. In fact they are subsequential limits of the following three subsequences, \( \{x_{3n}\} \), \( \{x_{3n-2}\} \), and \( \{x_{3n-1}\} \), respectively. These are the only subsequential limits, we conclude

\[
\lim \inf x_n = 0 \quad \text{and} \quad \lim \sup x_n = 2/3
\]

(b) Let \( x_n = (-1)^n(1 + 1/n) \).

Solution: Notice the subsequence of even terms, \( x_{2n} = 1 + 1/(2n) \), converges to \( 1 \) and the subsequence of odd terms, \( x_{2n-1} = -1 + 1/(2n-1) \), converges to \( -1 \). Can there exist a subsequential limit greater than \( 1 \) or less than \( -1 \)? No. (Why?)

\[
\lim \inf x_n = -1 \quad \text{and} \quad \lim \sup x_n = 1
\]

Solution Notes: You can also examine these two sequences at the definitional level (Definition 2.20) to arrive at the same conclusions.

Problem 19. Let \( \{r_n\} \) be a sequence consisting of all rational numbers in the open interval \((a, b)\). Calculate \( \lim \inf r_n \) and \( \lim \sup r_n \).

Proof: Claim \( \lim \inf r_n = a \) and \( \lim \sup r_n = b \). We prove only the first assertion. It is clear that \( a \) is a lower bound for \( \{r_n\} \). After reviewing Definition 2.20, we define

\[
a_n = \inf \{r_n \mid k \geq n\} \quad \forall n \in \mathbb{N}
\]

Claim that for all \( n \), \( a_n = a \). Indeed, \( a \) is a lower bound of \( \{r_n \mid k \geq n\} \), to show \( a \) is the infimum of this set, we take any number \( x > a \) and argue that \( x \) is not a lower bound of this set. We have \( a < \min \{x, b\} \), by Theorem 1.18, there is a rational number between these two numbers, if fact, as was commented on several times in
class, there are infinitely many rational number between \( a \) and \( \min\{x, b\} \). Note that the open intervals \((a, \min\{x, b\}) \) ⊆ \((a, b)\), so these rational numbers are within the interval \((a, b)\). Of these infinitely many rational numbers, choose one, denoted by \( r_k \), where \( k \geq n \). Thus \( r_k < \min\{x, b\} \leq x \). This shows that \( x \) is not a lower bound of the sequence \( \{r_n\} \), proves \( a_n = a \), and \( \lim \inf x_n = \lim a_n = a \). □

**Problem 22.** (Just prove the “lim sup” assertion, the other part is done similarly.)

**Proof:** We begin by proving a …

**Lemma.** For any nonempty set \( S \) that is bounded below, and for any number \( c \), \( \sup(c + S) = c + \sup S \).

**Proof:** Let \( \alpha = \sup(c + S) \) and \( \beta = \sup S \). Show \( \alpha = c + \beta \).

Let \( s \in S \), then \( \alpha \geq c + s \), thus, \( \alpha - c \geq s \). It follows that \( \alpha - c \) is an upper bound of \( S \), hence, \( \alpha - c \geq \sup S = \beta \). To show \( \alpha - c = \sup S \), let \( x < \alpha - c \), then \( x + c < \alpha \). But \( \alpha = \sup(c + S) \), so there is a \( s \in S \) such that \( c + s > x + c \). Thus, \( s > x \). This shows \( x \) is not an upper bound of \( S \). We have argued that \( \alpha - c \) is a upper bound of \( S \), and that any number \( x \) less than \( \alpha - c \) is not a upper bound of \( S \). Conclude \( \alpha - c = \sup S \), or \( \alpha = c + \beta \) and the lemma is proved. □

Now back to the problem at hand! After reviewing Definition 2.20,

\[
\limsup_{n \to \infty} (c + x_n) = \lim_{n \to \infty} \sup \{ c + x_k \mid k \geq n \} \quad \text{defn 2.20}
\]

\[
= \lim_{n \to \infty} \left( c + \sup \{ x_k \mid k \geq n \} \right) \quad \text{by the Lemma}
\]

\[
= c + \lim_{n \to \infty} \sup \{ x_k \mid k \geq n \}
\]

\[
= c + \limsup_{n \to \infty} x_n \quad \text{defn 2.20}
\]

**Problem 32.** Consider the sequences \( x_n = (-1)^n \) and \( y_n = (-1)^{n+1} \). Make some comparisons.

**Solution:** We have

\[
\lim \inf x_n = -1 \quad \lim \sup x_n = 1
\]

\[
\lim \inf y_n = -1 \quad \lim \sup y_n = 1
\]

Note that \( x_n + y_n = 0 \) and \( x_n y_n = -1, \forall n \in \mathbb{N} \). Thus,

\[
\lim \sup(x_n + y_n) = 0 \neq 2 = \lim \sup x_n + \lim \sup y_n
\]

\[
\lim \sup(x_n y_n) = -1 \neq 1 = \lim \sup x_n \cdot \lim \sup y_n
\]

Thus, the \( \lim \sup \) is not additive or multiplicative as \( \lim \) is. □