
Problem 2. Define \( x_n = \sum_{k=1}^{n} \frac{1}{n+k} \). Prove that \( \{x_n\} \) converges.

Proof: After making some sample calculations,

\[
x_1 = 0.5 \quad x_6 = 0.65321
\]
\[
x_2 = 0.58333 \quad x_6 = 0.65870
\]
\[
x_3 = 0.61666 \quad x_7 = 0.66287
\]
\[
x_4 = 0.63452 \quad x_8 = 0.66613
\]
\[
x_5 = 0.64563 \quad x_{10} = 0.66877
\]

we postulate that this sequence is increasing, and so, set out to prove it.

Now

\[
x_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \tag{1}
\]
\[
x_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \frac{1}{(n+1)+3} + \cdots + \frac{1}{2(n+1)}
\]
\[
= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2(n+1)}
\]
\[
= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2(n+1)}
\]
\[
= \left( \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} \right) + \frac{1}{2n+1} + \frac{1}{2(n+1)} \tag{2}
\]

and so, subtracting (1) from (2), we get

\[
x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2(n+1)}
\]
\[
= \frac{1}{2(2n+1)(n+1)} > 0
\]

Thus, for each \( n \in \mathbb{N} \), \( x_{n+1} - x_n > 0 \) or \( x_{n+1} > x_n \), that is the sequence \( \{x_n\} \) is strictly increasing.

To prove the sequence, it suffices to show the sequence is bounded above. This is obvious: first observe that \( \forall n, k \in \mathbb{N}, 1/(n+k) < 1/n \), and so

\[
x_n = \sum_{k=1}^{n} \frac{1}{n+k} < \sum_{k=1}^{n} \frac{1}{n} = 1
\]

This shows the sequence is bounded above, and hence converges. \( \square \)
Problem 3. (Similar to an example done in text.) Define $x_1 = 2$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n}$$

Prove that $\{x_n\}$ converges and find the limit of the sequence.

**Proof:** First, let us calculate a few terms:

- $x_1 = 2$
- $x_2 = 3.5$
- $x_3 = 3.178571$
- $x_4 = 3.162319$
- $x_5 = 3.162277$
- $x_6 = 3.162277$
- $x_7 = 3.162277$
- $x_8 = 3.162277$
- $x_9 = 3.162277$
- $x_{10} = 3.162277$

(Only six decimal points are shown above.) The sequence *appears* to be decreasing beginning with $n = 2$. Let’s begin by showing the sequence is bounded below.

Now, for $n \geq 2$, we have,

$$x_n^2 - 10 = \left(\frac{x_n}{2} + \frac{5}{x_n}\right)^2 - 10 = \frac{x_n^2}{4} + 5 + \frac{25}{x_n^2} - 10$$

$$= \frac{x_n^2}{4} - 5 + \frac{25}{x_n^2} = \left(\frac{x_n}{2} - \frac{5}{x_n}\right)^2 \geq 0$$

Thus

$$x_n^2 - 10 \geq 10$$

This shows $x_n^2 \geq 10$, $\forall n \geq 2$, or that $x_n \geq \sqrt{10}$, $\forall n \geq 2$.

Now show the sequence is decreasing, for $n \geq 2$.

$$x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n}$$

$$x_{n+1} - x_n = \frac{x_n}{2} + \frac{5}{x_n} - x_n = \frac{5}{x_n} - \frac{x_n}{2}$$

$$= \frac{10 - x_n^2}{2x_n}$$

From line (2) we have $x_n^2 - 10 \geq 0$, which, implies, by line (3) that $x_{n+1} - x_n \leq 0$, and the sequence is *decreasing*.

The sequence $\{x_n\}$ has been shown to be decreasing for $n \geq 2$. The sequence is bounded below by $\sqrt{10}$. We conclude $\{x_n\}$ converges.

To compute the limit, we set $L = \lim_{n \to \infty} x_n$ and reason as follows:

$$L = \lim_{n \to \infty} x_n = \lim_{n \to \infty} \left(\frac{x_n}{2} + \frac{5}{x_n}\right) = \frac{L}{2} + \frac{5}{L}$$
where we have liberally used the properties of convergent sequences, as listed in Theorem 2.7. The unknown limit satisfies the equation in line (4). Solving this equation we get \( L = \pm \sqrt{10} \). As our sequence is a sequence of nonnegative numbers, we conclude \( L = \sqrt{10} \). Thus, \( \lim_{n \to \infty} x_n = \sqrt{10} \). □

Problem 12. Show \( \{n/(n+3)\} \) is Cauchy.

Proof: Let \( x_n = n/(n+3) \), then for any \( n, m \in \mathbb{R} \), we have

\[
|x_m - x_n| = \left| \frac{m}{m+3} - \frac{n}{n+3} \right| = \frac{3|n - m|}{(n+3)(m+3)} \\
\leq \frac{3(n+m)}{(n+3)(m+3)} \leq \frac{3(n+m)}{nm} \\
\leq \frac{3}{m} + \frac{3}{n}
\]

(1)

Let \( \epsilon > 0 \), since \( \lim_{n \to \infty} (3/n) = 0 \), \( \exists N \) s.t. \( 3/n < \epsilon/2 \), \( \forall n \geq N \). Now, for \( n, m \geq N \), we have from

\[
|x_m - x_n| \leq \frac{3}{m} + \frac{3}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

and this completes the Cauchy argument. □

Problem 20. Give an example of a sequence \( \{x_n\} \) for which \( x_{n+1} - x_n \to 0 \), yet the sequence does not converge.

Proof: The easiest example is \( x_n = \ln(n) \). Then \( x_{n+1} - x_n = \ln(n+1) - \ln(n) = \ln((n+1)/n) \). It is clear that \( x_{n+1} - x_n \to 0 \), since \( (n+1)/n \to 1 \) and \( \ln 1 = 0 \). Finally, \( x_n \to \infty \). □

Solution Notes: Other examples: \( x_n = \sqrt{n} \), \( x_n = \sum_{k=1}^{n} \frac{1}{k} \).

Problem 24. (This is a very famous technique for showing a sequence is Cauchy, hence convergent. Try to get an useful upper bound on the expression \( |x_{n+k} - x_n| \). You should have a need to use Theorem 1.10.)

Proof: Given that \( |x_{n+1} - x_n| \leq r|x_n - x_{n-1}| \), where \( 0 \leq r < 1 \). First note that

\[
|x_{n+1} - x_n| \leq r^{n-1}|x_2 - x_1|
\]

(1)
is obtained by successively applying the given relation. Now,

\[ |x_{n+k} - x_n| \leq \sum_{j=1}^{k} |x_{n+j} - x_{n+j-1}| \] \quad \text{triangle inequality}

\[ \leq \sum_{j=1}^{k} r^{n+j-2} |x_2 - x_1| \] \quad \text{from (1)}

\[ \leq r^{n-1} \left( \sum_{j=1}^{k} r^{j-1} \right) |x_2 - x_1| \]

\[ \leq r^{n-1} \frac{1 - r^k}{1 - r} |x_2 - x_1| \] \quad \text{Theorem 1.10}

\[ \leq \frac{r^{n-1}}{1 - r} |x_2 - x_1| \]

We have shown that for any \( n, k \in \mathbb{N} \),

\[ |x_{n+k} - x_n| \leq \frac{r^{n-1}}{1 - r} |x_2 - x_1| \] \quad (2)

Let \( \epsilon > 0 \), since \( 0 \leq r < 1 \), it follows that \( \lim_{n \to \infty} r^{n-1} = 0 \), there exists a number \( N \) such that \( |r^{n-1}| < \epsilon \), \( \forall n \geq N \). Thus,

\[ |x_{n+k} - x_n| \leq \frac{r^{n-1}}{1 - r} |x_2 - x_1| \leq \frac{|x_2 - x_1|}{1 - r} \epsilon \quad \forall n \geq N \]

This is enough to prove that \( \{x_n\} \) is a Cauchy sequence. \( \square \)

**Problem 25.** (A simple application of Problem 24.) Let \( a_0 < a_1 \) and define \( a_n = (a_{n-1} + a_{n-2})/2 \), \( n > 1 \). Prove \( \{a_n\} \) is Cauchy and compute its limit.

**Proof:** It is easy to see

\[ |a_{n+1} - a_n| = \left| \frac{1}{2} (a_n + a_{n-1}) - a_n \right| = \frac{1}{2} |a_n - a_{n-1}| \]

or

\[ |a_{n+1} - a_n| \leq \frac{1}{2} |a_n - a_{n-1}| \]

By Problem 24, the sequence is Cauchy, hence convergent.

To compute the limit, we first note

\[ a_{n+1} - a_n = \frac{1}{2} (a_n + a_{n-1}) - a_n = \frac{1}{2} (a_{n-1} - a_n) = -\frac{1}{2} (a_n - a_{n-1}) \]

or

\[ a_{n+1} - a_n = -\frac{1}{2} (a_n - a_{n-1}) \] \quad (1)
From which it follows that
\[ a_{n+1} - a_n = \left( -\frac{1}{2} \right)^n (a_1 - a_0) \] (2)

Thus, from (2), we have expand as a finite telescoping sum
\[ a_n = a_0 + \sum_{k=0}^{n-1} (a_{k+1} - a_k) = a_0 + \sum_{k=0}^{n-1} \left( -\frac{1}{2} \right)^k (a_1 - a_0) \]
\[ = a_0 + \frac{1 - (-1/2)^n}{1 - (-1/2)} (a_1 - a_0) \]
\[ = a_0 + \frac{2}{3} (1 - (-1/2)^n)(a_1 - a_0) \] (2)

Taking the limit we get
\[
\lim_{n \to \infty} a_n = a_0 + \frac{2}{3} (a_1 - a_0) = \frac{1}{3} a_0 + \frac{2}{3} a_1
\]

and this completes the assignment! □