Problem 27. Let $A$ be a nonempty bounded set. The maximum of $A$ is a number $x \in A$ such that $a \leq x$, $\forall x \in A$. Prove that a nonempty bounded set has a maximum if and only if it contains its supremum.

Proof: ($\implies$) Suppose $A$ be a nonempty bounded set that has a maximum. Let $M = \max A$. Then, from the definition of maximum, $M$ is an upper bound of $A$. Now let $x < M$, then $x < M$, and $M \in A$. This shows that $x$ is not an upper bound of $A$. Hence, $M = \sup S$.

($\impliedby$) Now let $M = \sup A$ such that $M \in A$. Since $M$ is an upper bound of $A$ if follows that $x \leq M$, $\forall x \in A$. This fact, combined with the assumption that $M \in A$ shows that $M$ satisfies the definition of a maximum of $A$. □

Problem 11. Let $\{x_n\}$ be a sequence of real numbers.

(a) Suppose $\{x_n\}$ converges to $L$. Prove that $\{|x_n|\}$ converges to $|L|$.

Proof: The proof of this assertion follows from the Reverse Triangle Inequality:

$$||x_n| - |L|| \leq |x_n - L|$$

Details left to the reader. □

(b) Suppose $\{|x_n|\}$ converges. Show that $\{x_n\}$ may not converge.

Solution: Consider the sequence $x_n = (-1)^n$. Since $|x_n| = 1$, for all $x \in \mathbb{N}$, if follows that $\{|x_n|\}$ converges (to 1). But, as was proven in class, $\{x_n\} = \{(-1)^n\}$ does not converge. □

(c) Suppose $\{|x_n|\}$ converges to 0. Prove that $\{x_n\}$ converges to 0.

Proof: Suppose $\{|x_n|\}$ converges to 0. This means that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$||x_n| - 0| < \epsilon, \quad \forall n \geq N \quad (1)$$

But line (1) is equivalent to

$$|x_n| < \epsilon, \quad \forall n \geq N \quad (2)$$

since $||x_n|| = |x_n|$. This completes the proof, since (2) is the definition of the sequence $\{x_n\}$ converging to 0. □

Additional Solution Notes: Jared Hicks proved this using the Squeezing Theorem: For any $n \in \mathbb{N}$, he observed that $-|x_n| \leq x_n \leq |x_n|$, and $\lim -|x_n| = \lim |x_n| = 0$, Conclude therefore, $\lim x_n = 0$. □
Problem 13. Let \( \{a_n\} \) be a sequence that converges to \( L \) and let \( p \in \mathbb{N} \). Prove that the sequence \( \{a_{n+p}\} \) also converges to \( L \).

Proof: For simplicity and clarity of explanation, we define \( b_n = a_{n+p} \). Let \( \epsilon > 0 \), there exists a number \( N \in \mathbb{N} \) such that

\[
|a_n - L| < \epsilon, \quad \forall n \geq N
\]  

(1)

Now, for any \( n \geq N \), we know that \( n + p \geq N \) as well, since \( p \in \mathbb{N} \). It follows that for \( n \geq N \),

\[
|b_n - L| = |a_{n+p} - L| < \epsilon
\]  

(2)

But (2) is the definition of \( b_n \to L \). \( \Box \)

Problem 14. Let \( \{b_k\} \) be a sequence of nonnegative number converging to \( b > 0 \). Prove that \( \{\sqrt{b_k}\} \) converges to \( \sqrt{b} \). Prove the same result for the case \( b = 0 \).

Proof: Begin by supposing \( b > 0 \). For \( \epsilon > 0 \), there exists \( N \in \mathbb{R} \) such that

\[
|b_k - b| \leq \epsilon \sqrt{b}, \quad \forall k \geq N
\]  

(1)

Thus, for \( n \geq N \),

\[
|\sqrt{b_k} - \sqrt{b}| \leq \frac{|b_k - b|}{\sqrt{b_k + b}} \quad \text{rationalize}
\]

\[
\leq \frac{|b_k - b|}{\sqrt{b}} \quad \text{since} \quad \sqrt{b_k + b} \geq \sqrt{b}
\]

\[
< \frac{\epsilon \sqrt{b}}{\sqrt{b}} \quad \text{from (1)}
\]

\[
= \epsilon
\]

and this completes the proof when \( b > 0 \). \( \Box \)

Now assume \( b = 0 \). In this case, the proof is simple. Let \( \epsilon > 0 \) be given, choose \( N \in \mathbb{N} \) so that

\[
|b_k| < \epsilon^2, \quad \forall n \geq N
\]

Then, it follows,

\[
|\sqrt{b_k}| < \epsilon, \quad \forall n \geq N
\]

C’est tout. \( \Box \)

Problem 17. Suppose \( 0 < |y_k - x_k| < r_k, \forall k \in \mathbb{N} \), where \( r_k \to 0 \).

(a) Show that \( \{x_k\} \) and \( \{y_k\} \) need not converge.

Proof: Consider \( x_k = k + 1/(2k) \) and \( y_k = k \). Then \( 0 < |x_k - y_k| = 1/(2k) < 1/k = r_k \).

Neither of the sequences converge, since they are unbounded. \( \Box \)

(b) Suppose \( \{x_k\} \) converges to \( L \). Prove that \( \{y_k\} \) converges to \( L \), too.

Proof: The conclusion follows from the following inequality:

\[
|y_k - L| \leq |y_k - x_k| + |x_k - L| < r_k + |x_k - L|
\]  

(1)
For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
\[ r_k < \epsilon / 2, \quad |x_k - L| < \epsilon / 2, \quad \forall k \geq N \quad (2) \]

Now for $k \geq N$ we have from (1) and (2),
\[ |y_k - L| \leq |y_k - x_k| + |x_k - L| < r_k + |x_k - L| < \epsilon / 2 + \epsilon / 2 = \epsilon \]
and this completes the proof. \( \square \)

**Problem 18.** Suppose \( \{a_n\} \) converges to \( a > 0 \). Prove there exists a number \( m > 0 \) and a \( q \in \mathbb{N} \) such that \( a_n > m, \forall n \geq q \).

*Proof:* Take $\epsilon = a/2 > 0$, then since $a_n \to a$, there exists a number $q \in \mathbb{N}$ such that,
\[ |a_n - a| < a / 2, \quad \forall n \geq q \quad (1) \]

Playing around with the inequality in (1), we get
\[ a_n > a / 2, \quad \forall n \geq q \quad (2) \]

The proposition is proven, after putting that \( m = a / 2 \). \( \square \)