§1.1, pages 8–9.

**Problem 12.** This is the one I meant to assign on Assignment #1. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

*Solution:* Suppose $\sqrt{2} + \sqrt{3}$ is rational, then its square is rational as well. Put $r = (\sqrt{2} + \sqrt{3})^2 \in \mathbb{Q}$. Computing $r$ by squaring we get $r = 5 + 2\sqrt{6}$, and now solving for $\sqrt{6}$ in this expression, we see

$$\sqrt{6} = \frac{r - 5}{2}, \quad \text{where } r \in \mathbb{Q}$$

(1)

The right-hand side of (1) is clearly a rational number, we conclude that $\sqrt{6} \in \mathbb{Q}$, but this contradicts problem #11. □


**Problem 5.** Under what conditions on the numbers $a$ and $b$ does equality occur in the (Reverse) Triangle Inequality?

*Proof:* Claim $|a + b| = |a| + |b|$ if and only if $ab \geq 0$, that is, $a$ and $b$ have the same sign (or are zero). Indeed,

$$|a + b| = |a| + |b| \iff (|a + b|)^2 = (|a| + |b|)^2$$

$$\iff a^2 + 2ab + b^2 = a^2 + 2|ab| + b^2$$

$$\iff ab = |ab|$$

$$\iff ab \geq 0$$

Where have used the properties of absolute value without comment. The same condition is true for the Reverse Triangle Inequality, and the proof is the same. □

**Problem 6.** Suppose $|x - a| < \epsilon$, $|y - a| < \epsilon$. Compute $|x - y|$.

*Solution:* By the Triangle Inequality,

$$|x - y| = |(x - a) + (a - y)| \leq |x - a| + |a - y| \leq \epsilon + \epsilon = 2\epsilon$$

Thus $|x - y| < 2\epsilon$. □

**Problem 8(a).** Prove that for each $n > 1$,

$$\left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{n} |a_k|$$

(1)

*Hint:* Use Mathematical Induction (Appendix C). This is a generalize Triangle Inequality.
Proof: By the Triangle Inequality, (1) is true for the case \( n = 2 \).

Assume (1) is true for the \( n \in \mathbb{N} \), show it is true for \( n + 1 \). Indeed,

\[
\left| \sum_{k=1}^{n+1} a_k \right| = \left| \left( \sum_{k=1}^{n} a_k \right) + a_{n+1} \right|
\]

\[
\leq \left| \sum_{k=1}^{n} a_k \right| + |a_{n+1}| \quad \text{by Triangle Inequality}
\]

\[
\leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| \quad \text{by the induction assumption}
\]

\[
\leq \sum_{k=1}^{n+1} |a_k|
\]

and this completes the induction. \( \square \)

Problem 9. Let \( S \) be a nonempty set of real numbers and let \( a \) be a nonzero real number. Suppose \(|x-a| < |a|/2\) for all \( x \in S \). Prove that \(|x| \geq |a|/2, \forall x \in S\).

Proof: Let \( x \in S \), then by the Reverse Triangle Inequality and Theorem 1.4(a), we have

\[
|a|/2 > |x - a| \geq |x| - |a| \geq |a| - |x|
\]

Thus, \(|a| - |x| < |a|/2\). Transposing and combining appropriately, we have \(|x| > |a|/2\). \( \square \)

Problem 13. From the discussion at the bottom of page 12, we see that \( x \vee y = \max\{x, y\} \) and \( x \wedge y = \min\{x, y\} \).

Proof: WLOG, assume \( x \leq y \), so \( x \vee y = y \) and \( x \wedge y = x \). Note that since \( x \leq y \), it follows that \(|x - y| = y - x\). Thus

\[
\frac{x + y + |x - y|}{2} = \frac{x + y + y - x}{2} = \frac{2y}{2} = y = x \vee y
\]

\[
\frac{x + y - |x - y|}{2} = \frac{x + y - (y - x)}{2} = \frac{2x}{2} = x = x \wedge y
\]

and that completes the proof. \( \square \)

\( \S 1.3 \), pages 27–29.

Problem 1. Prove that a nonempty set is bounded if and only if it is both bounded above and bounded below.

Proof: ( \( \implies \) ) Let \( S \) be a nonempty set be bounded. Then there exists a number \( M \) such that

\[
|x| \leq M \quad \forall x \in S
\] (1)
But (1) is equivalent, by Theorem 1.4(a) to $-M \leq x \leq M$, $\forall x \in S$. Thus, we have

\begin{align*}
  x &\leq M \quad \forall x \in S \\
  x &\geq -M \quad \forall x \in S
\end{align*}

(2) \quad (3)

Line (2) is the definition of $S$ being bounded above, and (2) is the definition of $S$ being bounded below.

( $\iff$ ) Now assume $S$ is bounded above and below. We deduce there exists numbers $M$ and $m$ such that

\begin{align*}
  x &\leq M \quad \forall x \in S \\
  x &\geq m \quad \forall x \in S
\end{align*}

(4) \quad (5)

Define $K = \max\{|M|, |m|\}$. For any $x \in S$ we have,

\begin{equation}
  -K \leq m \leq x \leq M \leq K
\end{equation}

(6)

by Theorem 1.4(a), and the fact that $|M| \leq K$ and $|m| \leq K$. Once again we call on Theorem 1.4(a), to see that line (6) implies $|x| \leq K$, which is the defining relation for $S$ to be bounded. \(\square\)