Inverse Trigonometric Functions

In this section we will look at inverses of the six trigonometric functions and find their derivatives. Inverses of trigonometric functions are denoted using a -1 in a superscript (for example \( \cos^{-1} x \)) or by adding the prefix “arc” to the trigonometric function (for example \( \arccos x \)).

Because these functions are not 1-1 (i.e., they do not pass the horizontal line test), their inverses are not functions. Hence, we will restrict their domains so that they are 1-1 and the inverses on these restricted domains are functions.

Hence we want to satisfy the following conditions:

- The function has to be 1-1 on the restricted domain.
- The range of the function under the restricted domain has to be the same as the range of the function under its usual domain.

There are an infinite number of ways to restrict the domains to satisfy these two conditions. The most common inverses restrict the domains so that:

- The restricted domains include 0 when possible.
- The restricted domains contain the fewest intervals possible while still making sure that no range values are lost.

For example a restricted domain of \([0, \pi]\) satisfies all of these conditions for the cosine function. This gives us:

\[
\text{Restricted Domain} = D_f = [0, \pi] = R_{f^{-1}} \quad \text{and} \quad \text{Range} = R_f = [-1, 1] = D_{f^{-1}}
\]
Now we are going to sketch the graph of the inverse cosine function that corresponds to this restricted domain.

**Helpful Reminders:** When drawing an inverse of any function:
- If \((a, b)\) is on the graph of \(f\), then \((b, a)\) is on the graph of \(f^{-1}\). Sketch some of these to get started (especially intercepts).
- Remember that \(D_f = R_{f^{-1}}\) and \(R_f = D_{f^{-1}}\).
- Vertical asymptotes become horizontal asymptotes and vice-versa.
- Lightly sketch the line \(y = x\), because the graph of \(f^{-1}\) is the reflection of \(f\) over this line. Any points on the graph of \(f\) that fall on this line are also on the graph of \(f^{-1}\).

![Graphs](image)

**Particularly Helpful Information**

1. When solving problems involving this arccosine function, it is very helpful to remember that if \(\arccos x = \theta\), then \(\cos \theta = x\) for some \(0 \leq \theta \leq \pi\).

2. Simplifying compositions for this arccosine functions work this way:
   a. \(\arccos(\cos \theta) = \theta\) for \(0 \leq \theta \leq \pi\).
   b. \(\cos(\arccos x) = x\) for \(-1 \leq x \leq 1\).

3. If \(x \not\in [-1, 1]\), then \(\arccos x\) is undefined.

4. If \(\theta \not\in [0, \pi]\), then \(\cos \theta\) is defined, but \(\arccos(\cos \theta) \neq \theta\).
Example: Find (a) \( \arccos\left(-\frac{1}{2}\right) \) (b) \( \arccos\left(\cos\left(\frac{3\pi}{2}\right)\right) \) (c) \( \tan\left(\arccos\left(\frac{4}{5}\right)\right)\).

(a) When evaluating \( \arccos\left(-\frac{1}{2}\right) \), the key is to remember that
\[ \arccos\left(-\frac{1}{2}\right) = \theta \text{ where } \cos \theta = -\frac{1}{2} \text{ and } 0 \leq \theta \leq \pi. \] Thus, \( \theta = \frac{2\pi}{3} \).

(b) You may be tempted to assume that the functions will undo each other and you would get: \( \arccos\left(\cos\left(\frac{3\pi}{2}\right)\right) = \frac{3\pi}{2} \). However, the range of arccosine is \([0, \pi]\) and that does not include \( \frac{3\pi}{2} \). Here is one way to solve this problem:
\[ \arccos\left(\cos\left(\frac{3\pi}{2}\right)\right) = \arccos\left(0\right) = \theta \text{ where } \cos \theta = 0 \text{ and } 0 \leq \theta \leq \pi. \] Thus \( \theta = \pi/2 \).

You could solve this even if you did not know that \( \cos\left(\pi/2\right) = 0 \) using the method of the next example.

(c) When evaluating \( \tan\left(\arccos\left(\frac{4}{5}\right)\right) \) the key is to realize that \( \arccos\left(\frac{4}{5}\right) = \theta \text{ where } \theta \in [0, \pi] \) and \( \theta \) satisfies \( \cos \theta = 4/5 \).

Actually we must have \( \theta \in [0, \pi/2] \) because that is where cosine is positive.

The right triangle below has an angle with its vertex at the origin and a measure \( \theta \in [0, \pi/2] \). It has been drawn so that it satisfies \( \cos \theta = 4/5 \).

You can find that the length of the vertical leg is 3 by using Pythagorean's Formula.

Thus \( \tan\left(\arccos\left(4/5\right)\right) = \tan \theta = 3/4 \).
Differentiation

Remember, simplifying compositions for this arccosine functions work this way:

- \( \arccos(\cos \theta) = \theta \) for \( 0 \leq \theta \leq \pi \).
- \( \cos(\arccos x) = x \) for \( -1 \leq x \leq 1 \).

We can use implicit differentiation to find the derivative of this inverse function of cosine as follows:

Let \( y = \arccos x \). Then \( \cos y = x \) where \( 0 \leq y \leq \pi \). Differentiating both sides of \( \cos y = x \) with respect to \( x \) gives:

\[
(-\sin y) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}.
\]  

(1)

By the identity \( \sin^2 y + \cos^2 y = 1 \) we know that \( \sin y = \pm\sqrt{1-\cos^2 y} \).

But we know that \( \sin y \geq 0 \) because \( 0 \leq y \leq \pi \), thus \( \sin y = \sqrt{1-\cos^2 y} \).

Hence the formula in (1) can be written as:

\[
\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}.
\]  

(2)

Next, because \( \cos y = x \), equation (2) becomes:

\[
\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.
\]

Finally, because \( y = \arccos x \), we have just found that:

\[
\frac{d}{dx} (\arccos x) = -\frac{1}{\sqrt{1-x^2}}.
\]

Example Find the derivative of \( \cos^{-1}(e^{2x}) \).

\[
\frac{d}{dx} (\cos^{-1}(e^{2x})) = -\frac{1}{\sqrt{1-(e^{2x})^2}} \frac{d}{dx} (e^{2x}) = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}
\]
The derivatives of the other arctrig functions can be found similarly. Their derivatives appear below:

\[
\begin{align*}
\frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}}, \\
\frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}}, \\
\frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2}, \\
\frac{d}{dx}(\arccot x) &= -\frac{1}{1+x^2}, \\
\frac{d}{dx}(\text{arcsec} x) &= \frac{1}{x\sqrt{x^2-1}}, \\
\frac{d}{dx}(\text{arccsc} x) &= -\frac{1}{x\sqrt{x^2-1}}.
\end{align*}
\]

Now we will define and sketch an inverse for the other trigonometric functions.

Restricted \(D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = R_{f^{-1}}\) and \(R_f = [-1, 1] = D_{f^{-1}}\)

Restricted \(D_f = (-\frac{\pi}{2}, \frac{\pi}{2}) = R_{f^{-1}}\) and \(R_f = (-\infty, \infty) = D_{f^{-1}}\)

Note: The vertical asymptotes become horizontal asymptotes.
Restricted $D_f = (0, \pi) = R_{f^{-1}}$ and $R_f = (-\infty, \infty) = D_{f^{-1}}$

Note: The vertical asymptotes become horizontal asymptotes.

Restricted $D_f = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] = R_{f^{-1}}$ and $R_f = (-\infty, -1] \cup [1, \infty) = D_{f^{-1}}$

Note: The vertical asymptotes become horizontal asymptotes.

Restricted $D_f = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}] = R_{f^{-1}}$ and $R_f = (-\infty, -1] \cup [1, \infty) = D_{f^{-1}}$

Note: The vertical asymptotes become horizontal asymptotes.
When defining arcsecant, Stewart restricted the domain of secant to 
\([0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}]\) and when defining arccosecant he restricted the domain of cosecant to 
\((0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2})\).

One of the advantages of using his restrictions is that the ranges of the arcsecant and arccosecant have no negative values. Also, because the tangent function is never negative on 
\([0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}]\), this will make it possible to avoid absolute values when integrating some functions that contain the expression \(\sqrt{x^2 - 1}\). Stewart’s alternate inverses for the secant and sine function are shown below.

\[
D_f = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}], \quad R_f = (-\infty, -1] \cup [1, \infty) \quad D_f = (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}], \quad R_f = (-\infty, -1] \cup [1, \infty)
\]

Note: The vertical asymptotes become horizontal asymptotes.

**Example** Simplify \(\tan(\arcsin(e^x))\) where \(x < 0\).

To simplify \(\tan(\arcsin(e^x))\), where \(x < 0\), the key is to realize that \(\arcsin(e^x)\) is an angle \(\theta \in [-\pi/2, \pi/2]\) that satisfies \(\sin \theta = e^x\). When \(x < 0\) it happens that \(e^x \in (0,1)\), hence it is possible that \(\sin \theta = e^x\).

The right triangle above satisfies these conditions. Thus using Pythagorean’s formula we see that the length of the third leg is \(\sqrt{1-e^{2x}}\) and we have

\[
\tan(\arcsin(e^x)) = \tan \theta = \frac{e^x}{\sqrt{1-e^{2x}}}.\]
**Example** Find the derivative of \( f(x) = x \ln(\arctan x) \).

\[
f'(x) = 1 \cdot \ln(\arctan x) + x \cdot \frac{1}{\arctan x} \cdot \frac{1}{1 + x^2} = \ln(\arctan x) + \frac{x}{(1 + x^2) \arctan x}
\]

**Example** Evaluate \( \lim_{x \to 2} \frac{1}{x - 2} \).

As \( x \to 2^+ \), \( x - 2 \to 0^+ \), hence \( \frac{1}{x - 2} \to \infty \), which means that

\[
\arctan \left( \frac{1}{x - 2} \right) \to \frac{\pi}{2}.
\]