Graphing Review Part 3: Polynomials

Parabolas

Recall, that the graph of $y = f(x) = x^2$ is a parabola. It is an even function, hence it is symmetric about the $y$-axis. This means that $f(-x) = f(x)$. Its graph is shown below.

The point $(0,0)$ is called its vertex. From part 2 we learned that the graph of $y = (x-2)^2 + 1$ has the exact same shape as $y = x^2$, but it has been shifted 2 units to the right and 1 unit up. Hence, the vertex of this parabola will now be $(2,1)$. See the graph below on the left. From part 2 we also learned that the graph of $y = -\frac{1}{4}(x+1)^2 + 9$ will have a shape similar to $y = x^2$, but the vertex will be at $(-1,9)$ and because $0 < 1/4 < 1$ and $-1/4$ is negative, the graph will be wider and open downward. See the graph below on the right.

The **standard form** of a parabola is

$$y = a(x-h)^2 + k.$$
Example Find the specific equation of the parabola \( y = a(x-h)^2 + k \) with vertex \((2, 3)\) and that passes through \((1, 5)\).

The graph of \( x = y^2 \) will be a parabola that opens to the right and the graph of \( x = -y^2 \) will be a parabola that opens to the left. Note that they are not functions.

In this case, the **standard form** of a parabola is

\[
x = a(y-k)^2 + h.
\]
Example Sketch the region that is bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$. Be sure to include the coordinates of the points of intersection for these two graphs.

Example Sketch the region that is bounded between the graphs $y = x - 1$ and $y^2 = 2x + 6$. Be sure to include the coordinates of the points of intersection for these two graphs.
Consider the following three ways for writing the same quadratic function:

\[ y = -2(x - 3)^2 + 8 = -2x^2 + 12x - 10 = -2(x - 5)(x - 1) \]

1. In the first expression, you can immediately see that the vertex is \((3, 8)\), the graph opens downward, and is steeper than \(y = x^2\).

2. In the second expression, you can immediately see that the \(y\)-intercept is \((0, -10)\), the graph opens downward, and is steeper than \(y = x^2\). Also, because the tangent line to the graph is horizontal at the vertex, and the derivative (from calculus) will be 0 at the \(x\)-coordinate of the vertex, then you can find this \(x\)-coordinate by solving \(-4x + 12 = 0\) for \(x\). Because of this, the \(x\)-coordinate of the vertex for the equation \(y = ax^2 + bx + c\) is always \(x = -b/2a\).

3. In the third expression, you can immediately see that the \(x\)-intercepts are \((5, 0)\) and \((1, 0)\), the graph opens downward, and is steeper than \(y = x^2\). Also, due to symmetry, the \(x\)-coordinate of the vertex will be halfway between 5 and 1, so it will be 3.

**Example** Three parabolas are sketched below. Find their equations. You may assume that the coordinates of the vertices and intercepts are integers.
Polynomials of Higher Degree

When working with polynomials of higher degree, there are certain attributes that can help you to quickly sketch their graphs.

End Behavior
Consider the graphs of \( y = x^2 \), \( y = x^3 \), \( y = x^4 \), and \( y = x^5 \) shown below.

- They all have a \( y \)-intercept of 0.
- When \( x > 0 \), then \( y > 0 \) and when \( x < 0 \), then \( y > 0 \) for the even-degree functions and \( y < 0 \) for the odd-degree functions.
- When \( x > 1 \), then \( x^2 < x^3 < x^4 < x^5 \) and when \( 0 < x < 1 \) then \( x^5 < x^4 < x^3 < x^2 \) (see the graph above on the right).
- \( \lim_{x \to -\infty} y = \infty \) and \( \lim_{x \to -\infty} y = \begin{cases} \infty & \text{for the even-degree functions} \\ -\infty & \text{for the odd-degree functions} \end{cases} \)

Consider the graphs of \( y = -x^2 \), \( y = -x^3 \), \( y = -x^4 \), and \( y = -x^5 \) shown below.

\[ \lim_{x \to -\infty} y = -\infty \] and \( \lim_{x \to -\infty} y = \begin{cases} -\infty & \text{for the even-degree functions} \\ \infty & \text{for the odd-degree functions} \end{cases} \)
The behavior of a function as \( x \to \pm \infty \) is called its **end behavior**. In the case of polynomials, the end behavior can be generalized. Suppose that

\[
y = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0
\]

where \( a_n \neq 0 \), then when \( n \) is **even**

\[
\lim_{x \to \pm \infty} y = \begin{cases} 
\infty & \text{when } a_n > 0 \\
-\infty & \text{when } a_n < 0 
\end{cases}
\]

and when \( n \) is **odd**

\[
\lim_{x \to \pm \infty} y = \begin{cases} 
\infty & \text{when } a_n > 0 \\
-\infty & \text{when } a_n < 0 
\end{cases}
\]

To see why this is true, factor \( a_nx^n \) out of \( y = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0 \)

\[
y = a_nx^n \left( 1 + \frac{a_{n-1}}{a_nx^1} + \frac{a_{n-2}}{a_nx^2} + \ldots + \frac{a_3}{a_nx^3} + \frac{a_2}{a_nx^2} + \frac{a_1}{a_nx} + \frac{a_0}{a_nx^n} \right).
\]

The limit of the expression in parentheses is 1 as \( x \to \pm \infty \), hence as \( x \to \pm \infty \),

\( y \to a_nx^n \cdot 1 = a_nx^n \). This means that \( y = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0 \)

has the same end behavior as \( y = a_nx^n \).

Hence, a polynomial with a leading term of even degree has the same end behavior in each direction and a polynomial with a leading term of odd degree has the opposite end behaviors in the two directions.

**Example**

\[
\lim_{x \to -\infty} 4x + 800x^2 - 3x^3 = \quad \lim_{x \to -\infty} 4x + 800x^2 - 3x^3 =
\]

\[
\lim_{x \to -\infty} 4x + 800x^2 - 3x^4 = \quad \lim_{x \to -\infty} 4x + 800x^2 - 3x^4 =
\]
Example Three polynomials are sketched below. For each polynomial, determine whether the leading term has even or odd degree and whether the leading coefficient is positive or negative. The graphs show all of the places where directions and concavity are switched.

Behavior Near Roots
The graph of \( y = x^3 \left( \frac{1}{3} x - 1 \right) = \frac{1}{3} x^3 - x^2 \) is shown below. It has \( x \)-intercepts of \( (0,0) \) and \( (5,0) \). Because the polynomial has a factor of \( x^2 \), then the root \( x = 0 \) is a repeated zero of multiplicity 2 and because the polynomial has a factor of \( \left( \frac{1}{3} x - 1 \right) \), then the root \( x = 5 \) has multiplicity 1.

Notice that the graph bounces off the \( x \)-axis at \( (0,0) \) and passes through the \( x \)-axis at \( (5,0) \). The reasons for this are that \( x = 0 \) is a root of even multiplicity and \( x = 5 \) is a root of odd multiplicity:

- The function will not change signs immediately after \( x = 0 \) because \( x^2 \) does not.
- The function will change signs immediately after \( x = 5 \) because \( \left( \frac{1}{3} x - 1 \right)^3 \) does.
When we looked at the graphs of \( y = x^2, \ y = x^3, \ y = x^4, \) and \( y = x^5 \), we noticed that the graphs with higher degree were flatter (more horizontal) near \( x = 0 \).

A similar type of thing happens where the graphs of polynomials touch the \( x \)-axis; as the multiplicity of a root increases, the graph flattens near the root. Also, the slope of the tangent line to the polynomial will be zero at roots with multiplicity greater than 1. The graph of \( y = h(x) = -\frac{1}{20}(x + 3)^2(x + 1)(x - 1)^3(x - 3) \) is shown below. Notice how the graph is flat at \( x = -3 \) and \( x = 1 \) because these are roots with multiplicity greater than 1 and the graph is not as flat at \( x = -1 \) and \( x = 3 \) because these are roots with multiplicity 1. Also, because the multiplicity of \( x = -3 \) is even the graph bounces off the \( x \)-axis at \((-3, 0)\) and because the multiplicity of the roots \( x = -1, \ x = 1 \) and \( x = 3 \) are odd, the graph crosses through the \( x \)-axis at \((-1, 0), \ (1, 0) \) and \((3, 0)\).

**Careful:** It is possible for the graph of a polynomial to flatten near a root of multiplicity 1 and even look as though the slope of the tangent line is 0 at that root. Consider the polynomial \( g(x) = (x^3 + x) = x(x^2 + 1) \). It switches concavity at 0 and multiplying \( g(x) \) by \( 1/20 \) flattens the graph further; making it appear as though the multiplicity at 0 is greater than 1.
**Example** The polynomial sketched below has degree of either 9 or 10 (you will need to determine which). The absolute value of the leading term is 1/100. It has exactly 4 roots and they are all integers shown in the graph. Find the equation of this polynomial. The graph shows all of the places where directions and concavity are switched.

![Graph of a polynomial](image)

**Concavity**
A polynomial of degree \( n \geq 2 \) can switch its concavity at most \( n - 2 \) times. One way to see why this is true is to consider its second derivative. The second derivative will be a polynomial of degree \( n - 2 \). Hence, the second derivative has at most \( n - 2 \) roots.

This means that the second derivative can change sign (from positive to negative or from negative to positive) at most \( n - 2 \) times. Hence, the original polynomial can switch concavity (from concave up to concave down or from concave down to concave up) at most \( n - 2 \) times.

**Example** What is the minimum degree of the polynomial shown below?

![Graph of a polynomial](image)
**Strategy**

When graphing the polynomial

\[ y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - r_1)^{m_1} (x - r_2)^{m_2} \cdots (x - r_k)^{m_k}, \]

you can start by plotting the zeroes and remember that

- If the multiplicity of \( r_i \) is odd the graph will cross through the \( x \)-axis at \( x = r_i \) and if the multiplicity of \( r_j \) is even the graph will bounce off the \( x \)-axis at \( x = r_j \). The graph flattens out (becomes horizontal) at \( x = r_i \) as the multiplicity of this zero increases.
- The polynomial will have the same end behavior as \( y = a_n x^n \).
- The polynomial will switch concavity at most \( n - 2 \) times.
Solutions

Example Find the equation of the parabola \( y = a(x-h)^2 + k \) with vertex (2, 3) and that passes through (1, 5).

Because the vertex is (2, 3) the parabola has standard form \( y = a(x-2)^2 + 3 \). Because it passes through (1, 5):

\[
5 = a(1-2)^2 + 3 \implies 5 = a + 3 \implies a = 2
\]

\[
\Rightarrow y = 2(x-2)^2 + 3
\]

Example Sketch the region that is bounded by the parabolas \( y = 2x^2 \) and \( y = 1 + x^2 \). Be sure to include the coordinates of the points of intersection for these two graphs.

These are both parabolas that open upward and have vertices of (0,0) and (0,1), respectively. The graph of \( y = 2x^2 \) will be steeper than the graph of \( y = 1 + x^2 \), so even though \( y = 2x^2 \) is below \( y = 1 + x^2 \) when \( x = 0 \), the graphs will intersect. Their intersection can be found by setting them equal to each other:

\[
1 + x^2 = 2x^2 \implies 1 = x^2 \implies x = \pm 1
\]

Hence, they intersect at the points (-1, 2) and (1, 2). The region is shown below.
Example Sketch the region that is bounded between the graphs $y = x - 1$ and $y^2 = 2x + 6$. Be sure to include the coordinates of the points of intersection for these two graphs.

The graph of $y = x - 1$ is a line that has intercepts at $(0, -1)$ and $(1, 0)$.

The graph of $y^2 = 2x + 6 = 2(x + 3)$ will be a parabola that opens to the right with a vertex of $(-3, 0)$ and $y$-intercepts of $\left(0, \pm \sqrt{6}\right)$.

To solve for where they intersect, we can replace $y$ in the second equation with $x - 1$ (from the first equation) and we get:

$$(x - 1)^2 = 2x + 6 \implies x^2 - 2x + 1 = 2x + 6 \implies x^2 - 4x - 5 = 0 \implies (x + 1)(x - 5) = 0$$

Hence, they intersect when $x = -1$ and when $x = 5$: $(-1, -2)$ and $(5, 4)$.

The region is shown below.

![Graph showing the region bounded by the two graphs](image)

Example Three parabolas are sketched below. Find their equations. You may assume that the coordinates of the vertices and intercepts are integers.

The graph of $y = f(x)$ has a vertex at $(3, 0)$ and it opens upward. So, it has the form $y = a(x - 3)^2$ where $a > 0$. Because it has a $y$-intercept of $(0, 3)$, then $3 = a(0 - 3)^2 \implies 3 = 9a \implies a = 1/3$. Hence, $f(x) = \frac{1}{3}(x - 3)^2$.

The graph of $y = g(x)$ has $x$-intercepts of $(-3, 0)$ and $(5, 0)$, and the graph opens downward. So, it has the form $y = a(x + 3)(x - 5)$ where $a < 0$. Because it has a $y$-intercept of $(0, 30)$, then $30 = a(0 + 3)(0 - 5) \implies 30 = -15a \implies a = -2$. Hence, $g(x) = -2(x + 3)(x - 5)$. Notice, it was not necessary to use its vertex of $(1, 32)$. 
The graph of $y = h(x)$ has a vertex at $(-4, -1)$, a $y$-intercept of $(0, -9)$, and it opens downward. So, it has the form $y = a(x + 4)^2 - 1$ where $a < 0$. Because it has a $y$-intercept of $(0, -9)$, then 

$$-9 = a(0 + 4)^2 - 1 \Rightarrow -8 = 16a \Rightarrow a = -\frac{1}{2}.$$ Hence, $h(x) = -\frac{1}{2} (x + 4)^2 - 1$.

**Example**

$$\lim_{x \to \infty} 4x + 800x^2 - 3x^3 = -\infty \quad \lim_{x \to -\infty} 4x + 800x^2 - 3x^3 = \infty$$

$$\lim_{x \to \infty} 4x + 800x^2 - 3x^4 = -\infty \quad \lim_{x \to -\infty} 4x + 800x^2 - 3x^4 = -\infty$$

**Example** Three polynomials are sketched below. For each polynomial, determine whether the leading term has even or odd degree and whether the leading coefficient is positive or negative.

- $y = f(x)$: Odd, Positive
- $y = g(x)$: Even, Negative
- $y = h(x)$: Odd, Negative

**Example** The polynomial sketched below has degree of either 9 or 10 (you will need to determine which). The absolute value of the leading term is $1/100$. It has exactly 4 roots and they are all integers shown in the graph. Find the equation of this polynomial.
Because the graph has the same end behavior in both directions, it has even degree; hence \( n = 10 \). Because it opens downward, the leading coefficient is negative; hence it is \(-1/100\).

The only possible roots are \( x = -3, \ x = -1, \ x = 0 \) and \( x = 2 \).

- Because the graph flattens out and crosses through the \( x \)-axis at \( x = -3 \), then the multiplicity of this root is odd and greater than 1; hence it is at least 3.
- Because the graph does not look as though it flattens out at \( x = -1 \) but does cross through the \( x \)-axis at \( x = -1 \), then the multiplicity of this root is odd and probably equal to 1.
- Because the graph flattens out at \( x = 0 \) and appears to bounce off of the \( x \)-axis at \( x = 0 \), then the multiplicity of this root is even and greater than 1; hence it is at least 2.
- Because the graph flattens out at \( x = 2 \) and appears to bounce off of the \( x \)-axis at \( x = 2 \), then the multiplicity of this root is even and greater than 1; hence it is at least 2.

If you add up the minimum multiplicities you get \( 3 + 1 + 2 + 2 = 8 \). This cannot be correct because we know the degree is 10, hence one of these needs to increase by 2. Because the graph looks flattest at \( x = 0 \), I would guess that that root has the highest multiplicity. This gives me a polynomial of:

\[
y = f(x) = -\frac{1}{100}(x + 3)^3(x + 1)^4(x - 2)^2
\]

You could plot some points to see that this is the only choice that is reasonable under the given constraints.

**Example** What is the minimum degree of the polynomial shown below?

The graph looks like it switches concavity at about \( x = -2.8, \ x = -1.2, \ x = 0.3, \ x = 1, \) and \( x = 2 \). If it switches concavity 5 times, its minimum degree is 7.