Graphing Review Part 1: Circles, Ellipses and Lines

Definition

The graph of an equation is the set of ordered pairs, \((x, y)\), that satisfy the equation.

We can represent the graph of a function by sketching the set of ordered pairs, \((x, y)\), that satisfy the equation on the coordinate plane.

A sketch of the graph of \(y = \frac{2(x-2)}{x-1}\) is shown below. The points \((0, 4)\), \((2, 0)\) and \((5, 1.5)\) lie on the graph because they satisfy the equation (i.e., make it true). You can check this by replacing the \(x\)- and \(y\)-values in the equation with these coordinates:

\[
4 = \frac{2(0-2)}{0-1} \quad 0 = \frac{2(2-2)}{2-1} \quad 1.5 = \frac{2(5-2)}{5-1}.
\]

The point \((4, 7)\) does not lie on the graph because it does not make the equation true

\[
7 \neq \frac{2(4-2)}{4-1}.
\]

When graphing an equation, one basic strategy is to plot points on a coordinate system, try to determine any patterns, and then connect the points appropriately.

To improve on this, we are going to explore the graphs of different types of equations and use what we find to guide us when drawing unfamiliar equations.
Equations of Circles

The first type of equation we are going to examine has the form \((x-h)^2 + (y-k)^2 = r^2\) where \(h\), \(k\), and \(r\) are constants and where \(r > 0\). The graph of this equation will be a circle with center \((h,k)\) and radius \(r\). To help see why this is true, we are going to start by looking at the equation for the distance between 2 points.

We will start by sketching the points \((-4,4)\) and \((2,-5)\) and the line segment that connects them (see the figure below on the left). The distance between these points is the length of this line segment.

Next, we will form a right triangle that has this line segment as its hypotenuse by drawing a vertical line segment from \((-4,4)\) to \((-4,-5)\) and then a horizontal line segment from \((-4,-5)\) to \((2,-5)\) (see the figure above on the right).

Now we can use Pythagorean’s formula to calculate the length of the hypotenuse of this triangle. Note: The length of the hypotenuse is the distance between \((-4,4)\) and \((2,-5)\).

The length of the vertical side is \(4 - (-5) = 9\) and the length of the horizontal side is \(2 - (-4) = 6\). If we label the length of the hypotenuse as \(c\) we have:

\[
9^2 + 6^2 = c^2 \quad \Rightarrow \quad 81 + 36 = c^2 \quad \Rightarrow \quad 117 = c^2 \quad \Rightarrow \quad c = \sqrt{117}
\]

If these points are relabeled as \((x_1, y_1)\) and \((x_2, y_2)\) the formula for the distance between these points looks like:

\[
D((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]
Notice: It does not matter in which order you subtract the coordinates because the difference is squared. I usually order it so that the differences are non-negative:

\[ \sqrt{((\text{larger } x\text{-value}) - (\text{smaller } x\text{-value}))^2 + ((\text{larger } y\text{-value}) - (\text{smaller } y\text{-value}))^2}. \]

Next, suppose the distance between the point \((x, y)\) and the point \((h, k)\) is \(r\); the distance formula gives us:

\[ \sqrt{(x-h)^2 + (y-k)^2} = r \]

Squaring both sides of this we have:

\[ (x-h)^2 + (y-k)^2 = r^2. \]

Notice, this is an equation and if \((h, k)\) is a fixed point, then the set of points, \((x, y)\), that satisfy this equation are exactly the points that are a distance of \(r\) from \((h, k)\). Hence the graph of this equation is the circle with center \((h, k)\) and radius \(r\).

Note: If you have a version of this handout in color, the radius and center are shown in red because they are not part of the circle.

Suppose we take the equation \((x-h)^2 + (y-k)^2 = r^2\) and solve for the variable \(y\):

\[ \Rightarrow (y-k)^2 = r^2 - (x-h)^2 \]

\[ \Rightarrow y-k = \pm \sqrt{r^2 - (x-h)^2} \]

\[ \Rightarrow y = k \pm \sqrt{r^2 - (x-h)^2}. \]

The graph of \(y = k + \sqrt{r^2 - (x-h)^2}\) will give us the points on the circle with \(y\)-coordinates greater than or equal to \(k\) (the upper half of the circle) and the graph of \(y = k - \sqrt{r^2 - (x-h)^2}\) will give us the points on the circle with \(y\)-coordinates less than or equal to \(k\) (the lower half of the circle).
If we solved the equation \((x-h)^2 + (y-k)^2 = r^2\) for the variable \(x\) we would get:

\[x = h \pm \sqrt{r^2 - (y-k)^2}.\]

What is the graph of \(x = h + \sqrt{r^2 - (y-k)^2}\)?
(Note: This is not the equation of a function.)

This will give us the points on the circle with \(x\)-coordinates greater than or equal to \(h\) (the right half of the circle).

What is the graph of \(x = h - \sqrt{r^2 - (y-k)^2}\)?
(Note: This is not the equation of a function.)

This will give us the points on the circle with \(x\)-coordinates less than or equal to \(h\) (the left half of the circle).

**Examples**

1. Find the equation of a circle with center \((5, -8)\) and radius 10.

2. Find the equation of the circle shown below. For ease of computation, you may assume that the coordinates of the center and the radius are integers.

3. Find the center and radius of the following circles.
   
   a. \((x+2)^2 + (y-7)^2 = 3\)
   b. \(x^2 - 6x + y^2 + 4y = 3\)
   c. \(x^2 + y^2 = 1\)
4. Sketch the graph of the following equations.

a. \( y = 5 + \sqrt{9 - (x + 2)^2} \)

b. \( y = 3 - \sqrt{1 - x^2} \)

c. \( x = \sqrt{5 - y^2} - 3 \) \text{ Note: This is not the equation of a function.}

d. \( x = -2 - \sqrt{4 - y^2} \) \text{ Note: This is not the equation of a function.}
Equations of Ellipses

Notice, if we divide both sides of the equation of the circle \( x^2 + y^2 = 9 \) by 9 we get:

\[
\frac{x^2}{3^2} + \frac{y^2}{3^2} = 1.
\]

How would the graph change if one of the threes was changed to a five, as shown below?

\[
\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1
\]

This equation is called an ellipse. It has an oval-like shape. To sketch it, notice that when \( y = 0 \), then \( x = \pm 3 \) and when \( x = 0 \), then \( y = \pm 5 \). Now you have four points that you can plot. Then connect these to get the graph as shown below.

As with the equations of circles, an ellipse with a center of \( (h,k) \) can be written as

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

where \( a \) and \( b \) are distinct positive constants (because if \( a=b \) then we have an equation of a circle). **Note:** Because \( a \neq b \), we usually let the variable \( a \) be the larger of the two.
Examples

1. Sketch the graph of the following equations.

   a. \[ y = 5 + 2\sqrt{1 - (x + 2)^2 / 36} \]

   b. \[ x = -2 - 3\sqrt{1 - \frac{y^2}{4}} \] \textbf{Note}: This is not the equation of a function.
Equations of Lines

The graph below shows the line through the points (3, 2) and (6, 10).

As shown in the graph, if you start at the point (3, 2) and move 8 units vertically upward (the $y$-coordinate increases by 8) followed by 3 units horizontally to the right (the $x$-coordinate increases by 3), then you will end up at the point (6, 10); the change in the $y$-coordinate is 8 and the change in the $x$-coordinate is 3.

The ratio of these two numbers, $8/3$, is called the \textbf{slope} of the line. The slope tells us about how the line is slanted (i.e., sloped). We usually use the letter $m$ to denote the slope of a line.

Because the vertical change is in the numerator and the horizontal change is in the denominator, the slope is often referred to as the \textbf{rise over the run}.

You will get the same slope value, $8/3$, regardless of which point you start at and regardless of which 2 distinct points you choose on this line.

- To see why you get the same slope regardless of which point you start at, then start at the point (6, 10) and move 8 units vertically downward (the $y$-coordinate decreases by 8) followed by 3 units horizontally to the left (the $x$-coordinate decreases by 3), then you will end up at the point (3, 2); the change in the $y$-coordinate is $-8$ and the change in the $x$-coordinate is $-3$. The ratio of these two numbers, $(-8)/(-3)$. This gives the same slope value because $(-8)/(-3) = 8/3$. 
To see why you would get the same slope if you had chosen any pair of distinct points \((x_1, y_1)\) and \((x_2, y_2)\), consider the graph below. Each pair of points could be associated with a right triangle, as shown. All of these right triangles are similar to each other. Hence, the ratio of the lengths of their vertical sides divided by their horizontal sides must all be the same. Because this ratio is \(8/3\) for one of the triangles, it will \(8/3\) for all of the triangles.

![Graph showing two points and a right triangle]

For the points \((x_1, y_1)\) and \((x_2, y_2)\), the ratio will be given by

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{8}{3}.
\]

We are going to use this ratio to find the equation of the line. Suppose that \((x, y)\) is any point on this line other than \((3, 2)\). Because \((x, y)\) and \((3, 2)\) are distinct points on the line, then we know that:

\[
\frac{y - 2}{x - 3} = \frac{8}{3}, \quad (1)
\]

This is an equation and we know that the equation holds for any point \((x, y)\) on the line except \((3, 2)\). Why is it that equation (1) is not true when \((x, y) = (3, 2)\)?

It is not defined when \(x = 3\) because you cannot have a 0 as a denominator.

If we multiply both sides of (2) by \((x - 3)\) we get the equation:

\[
y - 2 = \frac{8}{3} (x - 3).
\]

This equation will also be true when \((x, y) = (3, 2)\). Hence, if you want to graph this equation, you need to include all of the points on the line.
If the points on the line are the only points that make this equation true, then the equation of this line is

$$y - 2 = \frac{8}{3}(x - 3).$$

How do we know that the points, \((x, y)\), that are not on the line do not satisfy this equation? The key is that when \((x, y)\) is not on the line, the ratio

$$\frac{y - 2}{x - 3} = \frac{8}{3}$$

will not hold. Hence, the points that are not on the line do not satisfy the equation

$$y - 2 = \frac{8}{3}(x - 3).$$

The graphs below show some of the cases where \((x, y)\) is not on the line. For each of these cases, a right triangle has been drawn that includes the point \((3, 2)\).

- In the left case the triangles are not similar.
- In the middle case you get the proportion \(\frac{y - 2}{3 - x} = \frac{8}{3}\) and in the right case you get the proportion \(\frac{2 - y}{x - 3} = \frac{8}{3}\), but both of these are equivalent to \(\frac{y - 2}{x - 3} = -\frac{8}{3}\).

**THE BIG IDEA:** The only points that satisfy the equation

$$y - 2 = \frac{8}{3}(x - 3)$$

are exactly the points on the line. Hence, that is the equation of the line.
When looking at the equation

\[ y - 2 = \frac{8}{3}(x - 3) \]

it can be immediately observed that the point \((3, 2)\) satisfies the equation and the slope of the line is \(8/3\). That is why this form is called **point-slope form**. In general this looks like:

\[ y - y_1 = m(x - x_1) \]

where \((x_1, y_1)\) is the point and \(m\) is the slope. This is handy to use when you need to find the equation of a line and you know a point on it and its slope or when you know two points on it.

**Example**

Use point-slope form to find the equation of the line through \((5, -4)\) that has a slope of \(m = 2\).

**Example**

Use point-slope form to find the equation of the line through \((5, -4)\) and \((-2, 1)\).

Often, these equations are solved for \(y\) and simplified. For our original equation this would look like:

\[
\begin{align*}
y - 2 &= \frac{8}{3}(x - 3) \\
\quad &\iff y = \frac{8}{3}x - \frac{14}{3} \\
\quad &\iff y = \frac{8}{3}x - 6
\end{align*}
\]

In general this is called **slope-intercept form**. Why is that a good name for it?

Because just looking at it you can immediately see its slope, \(8/3\), and its \(y\)-intercept, \((0, -6)\).

Another form for a line is called **standard form** (or **general form**). A line is in standard form when written as \(ax + by + c = 0\). This form is particularly helpful when the slope is undefined or when working with lines in 3-dimensional space.
Example
Write the line $y = \frac{2}{3}x - 6$ in standard form.

Note that we can multiply both sides of this by 3 and it will still be in standard form:

$$8x - 3y - 18 = 0$$

Example What is the slope of the line $4y - 5x + 9 = 3$

Example Find the equations of the lines below. Each line has at least one pair of integer coordinates. For example, line D contains the point $(3, -5)$. 

![Graph of lines A, B, C, and D](image)
Example Find the equations of the three lines that contain the sides of the triangle with vertices $\text{(2,0)}$, $\text{(0,1)}$ and $\text{(2,1)}$.

Example Sketch the region bounded between the lines $y = -\frac{1}{2} x + 1$, $y = \frac{1}{2} x$, and $x = 0$. 
**Parallel and Perpendicular Lines**

If two lines are parallel, what is true about their slopes?

They are the same

When two lines are perpendicular, if one of them has a nonzero slope of \( m \) then the line perpendicular to it has a slope of \( -\frac{1}{m} \).

Looking at the perpendicular lines below, it makes sense that if one of them has a positive slope, then the other would have a negative slope. Also, if one had a steep slope, you would expect the other to have a flatter slope. But why does this formula work? Let’s look at a proof.

**Explanation**

Call the lines \( L_1 \) and \( L_2 \). Because the lines are not parallel, they will intersect at some point, say \((a, b)\).

If the slope of \( L_1 \) is \( m_1 \), then if we start at the point \((a, b)\) on \( L_1 \) and I travel \( m_1 \) vertically followed by 1 horizontally, we will land on the point \((a+1, b + m_1)\).

This point is on \( L_1 \) because the rise and run that we just traveled from \((a, b)\) has the quotient \( \frac{m_1}{1} = m_1 \).

Likewise, if the slope of \( L_2 \) is \( m_2 \) then the point \((a+1, b + m_2)\) is on \( L_2 \).

Next, connect the points \((a+1, b + m_1)\) and \((a+1, b + m_2)\) with a line segment. We have just formed a right triangle, thus we can use Pythagorean’s formula and the distance formula to get:

\[
(1^2 + m_1^2) + (1^2 + m_2^2) = (m_1 - m_2)^2 \Rightarrow 2 + m_1^2 + m_2^2 = m_1^2 - 2m_1 \cdot m_2 + m_2^2
\]

\[
\Rightarrow -1 = m_1 \cdot m_2 \Rightarrow m_1 = -\frac{1}{m_2}
\]
Solutions

1. Find the equation of a circle with center \((5, -8)\) and radius 10.

\[(x - 5)^2 + (y + 8)^2 = 100\]

2. Find the equation of the circle shown below. For ease of computation, you may assume that the coordinates of the center are integers.

\[(h, k) = (-4, 3) \quad r = 5 \quad \Rightarrow (x + 4)^2 + (y - 2)^2 = 5^2\]

3. Find the center and radius of the following circles.
   a. \((x + 2)^2 + (y - 7)^2 = 3\) \hspace{1cm} \(h, k = (-2, 7)\) \hspace{1cm} \(r = \sqrt{3}\)
   b. \(x^2 - 6x + y^2 + 4y = 3\)
      Complete the squares:
      \[x^2 - 6x + 9 + y^2 + 4y + 4 = 3 + 9 + 4\]
      \[\Rightarrow (x - 3)^2 + (y + 2)^2 = 16 \quad \Rightarrow (x - 3)^2 + (y + 2)^2 = 4^2\]
      \((h, k) = (3, -2)\) \hspace{1cm} \(r = 4\)
   c. \(x^2 + y^2 = 1\)
      \[(x - 0)^2 + (y - 0)^2 = 1^2 \quad \Rightarrow (h, k) = (0, 0) \quad r = 1\]

4. Sketch the graph of the following equations.
   a. \[y = 5 + \sqrt{9 - (x + 2)^2}\]

   Notice that the \(y\)-coordinates will be greater than or equal to 5 (so this will be the upper half of the circle). To make it easier to determine the center and radius of the circle, we will put the equation in standard form:
b. \( y = 3 - \sqrt{1-x^2} \)

Notice that the \( y \)-coordinates will be less than or equal to 3 (so this will be the lower half of the circle). To make it easier to determine the center and radius of the circle, we will put the equation in standard form:

\[
\sqrt{1-x^2} = 3 - y = \Rightarrow 1-x^2 = (3-y)^2 = 1=x^2+(y-3)^2
\]

\((h, k) = (0, 3)\) \( r = 1 \) \( \text{Remember, } y \leq 3 \).
d. \[ x = -2 - \sqrt{4 - y^2} \]  \hspace{1cm} \textbf{Note:} This is not the equation of a function.

The \( x \)-coordinates are less than or equal to \(-2\) (the left half of the circle).

\[ x = -2 - \sqrt{4 - y^2} \Rightarrow x + 2 = -\sqrt{4 - y^2} \Rightarrow (x + 2)^2 = 4 - y^2 \Rightarrow (x + 2)^2 + y^2 = 4 \]

\((h, k) = (-2, 0)\) \hspace{1cm} \(r = 2\) \hspace{1cm} \text{Remember, } x \leq -2.\]

\[\begin{array}{c}
\text{Ellipse Examples} \\
1. \text{Sketch the graph of the following equations.} \\
a. \quad y = 5 + 2\sqrt{1 - (x + 2)^2 / 36} \\
\text{Notice that the } y \text{-coordinates will be greater than or equal to } 5 \text{ (so this will be the upper half of the ellipse). To make it easier to determine the center of the ellipse, we will put the equation in standard form:} \\
\quad y - 5 = 2\sqrt{1 - (x + 2)^2 / 36} \Rightarrow \frac{(y - 5)^2}{2^2} = 1 - \frac{(x + 2)^2}{6^2} \Rightarrow \frac{(x + 2)^2}{6^2} + \frac{(y - 5)^2}{2^2} = 1 \\
\quad (h, k) = (-2, 5) \hspace{1cm} \text{Remember, } y \geq 5. \\
\end{array}\]
b. $x = -2 - 3\sqrt{1 - \frac{y^2}{4}}$ \hspace{1cm} \textbf{Note}: This is not the equation of a function.

The $x$-coordinates are less than or equal to $-2$ (the left half of the ellipse). To make it easier to determine the center of the ellipse, we will put the equation in standard form:

$x + 2 = -3\sqrt{1 - \frac{y^2}{4}} \Rightarrow (x + 2)^2 = 9\left(1 - \frac{y^2}{4}\right) \Rightarrow \frac{(x + 2)^2}{9} = 1 - \frac{y^2}{4} \Rightarrow \frac{(x + 2)^2}{9} + \frac{y^2}{4} = 1$

$(h, k) = (-2, 0) \quad \text{Remember, } x \leq -2.$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ellipse_graph.png}
\caption{Graph of the ellipse}
\end{figure}

**Equations of Lines Solutions**

**Example**

Use point-slope form to find the equation of the line through $(5, -4)$ that has a slope of $m = 2$.

$y - (-4) = 2(x - 5) \Rightarrow y + 4 = 2(x - 5)$

**Example**

Use point-slope form to find the equation of the line through $(5, -4)$ and $(-2, 1)$.

First we will find the slope: $m = \frac{\text{rise}}{\text{run}} = \frac{1 - (-4)}{-2 - 5} = \frac{5}{-7} = \frac{-5}{7}$. Thus the equation of the line is:

$y - (-4) = \frac{-5}{7}(x - 5) \Rightarrow y + 4 = \frac{-5}{7}(x - 5)$

**Example**

Write the line $y = \frac{5}{7}x - 6$ in standard form.

$\frac{5}{7}x - y - 6 = 0$
Note that we can multiply both sides of this by 3 and it will still be in standard form:

$$8x - 3y - 18 = 0$$

**Example** What is the slope of the line $4y - 5x + 9 = 3$?

One way to solve this is by putting it in slope-intercept form: $y = \frac{5}{4}x - \frac{3}{2}$.

Thus the slope is $\frac{5}{4}$.

**Example** Find the equations of the lines below.

If you tried to compute the slope of A, you wind up with a $0$ in the denominator. So the slope of A is undefined. The line A contains any point with an $x$-coordinate of $-3$. The solutions of the equation $x = -3$ will be any points with an $x$-coordinate of $-3$, hence the equation is $x = -3$.

The slope of B is $0$ and it has a y-intercept of $2$, thus $y = 2$.

The line C passes through $(0, -2)$ and $(4, 1)$, so its slope is $\frac{3}{4}$ which means its equation is $y = (3/4)x - 2$.

The line D passes through $(-4, -1)$ and $(3, -5)$, so its slope is $-\frac{4}{7}$.

Using $(-4, -1)$ gives us:

$$y - (-1) = -\frac{4}{7}(x - (-4)) \Rightarrow y = -\frac{4}{7}(x + 4) - 1 = -\frac{4}{7}x - \frac{16}{7} - 1 = -\frac{4}{7}x - \frac{23}{7}$$
**Example** Find the equations of the three lines that contain the sides of the triangle with vertices (2,0), (0,1) and (2,1).

\[ y = 1, \quad y = -\frac{1}{2}x + 1, \quad x = 2 \]

![Diagram of a triangle with equations](image)

**Example** Sketch the region bounded between the lines \[ y = -\frac{1}{2}x + 1, \quad y = \frac{1}{2}x, \]
and \[ x = 0. \]

![Diagram of a region bounded by lines](image)