ANALYSIS OF THE GENERALIZED CATALAN ORBITS

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Brittany Nicole Mott

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ANALYSIS OF THE GENERALIZED CATALAN ORBITS

Brittany Nicole Mott

Thesis

Approved:  Accepted:

Advisor  Dean of the College
Dr. James P. Cossey  Dr. Chand Midha

Faculty Reader  Dean of the Graduate School
Dr. Jeffrey Riedl  Dr. George R. Newkome

Faculty Reader  Date
Dr. Stefan Forcey

Department Chair  
Dr. Timothy Norfolk
ABSTRACT

The Catalan numbers are well-known in the field of combinatorics. This sequence of integers counts a variety of combinatorial objects including binary trees, triangulations of regular polygons, and paths on a square grid which lie below the diagonal. Of the hundreds of realizations of the Catalan numbers, the majority can be extended and considered under the scope of the generalized Catalan numbers. The generalized Catalan numbers consider a second parameter in addition to the single parameter of the regular Catalan numbers, consequently forming a grid of values instead of a single sequence of integers. Although a formula has been developed to determine the generalized Catalan number when given values for the two parameters, much is still unknown about the generalized Catalan numbers. This thesis will count the generalized Catalan orbits, defined to be the orbits of the combinatorial objects counted by the generalized Catalan numbers. Two realizations of the generalized Catalan numbers, \( p \)-ary trees and the dissection of regular polygons, will be used throughout our study of the generalized Catalan orbits. A variety of techniques will be used to create formulas which describe the number of generalized Catalan orbits in terms of the given parameters.
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CHAPTER I
INTRODUCTION

This thesis will examine the properties of the orbits of the combinatorial objects related to the generalized Catalan numbers, $C_{p,k}$. The Catalan numbers, $C_n$, form a sequence of positive integers that is well-known and occurs frequently in many aspects of the field of combinatorics. The generalized Catalan numbers consider a parameter additional to the single parameter considered in the regular Catalan case. The generalized Catalan numbers are more flexible, creating a number using a pair of values $p$ and $k$, where $p, k \in \mathbb{N}$. The formula to determine the $n$th Catalan number given a value of $n$ is well known, as is the formula to determine the generalized Catalan number given a pair, $(p, k)$. We wish to expand our knowledge of the Catalan numbers by considering the number of orbits for the combinatorial objects counted by $C_{p,k}$ under a natural geometric action. Our ultimate goal is to analyze the number of orbits of these geometric objects and produce a formula that gives the number of orbits in terms of the given pair $(p, k)$.

We will approach this goal by considering Table 1 which displays the number of orbits for some small values of $p$ and $k$. Values in the table are calculated using a program that was developed to algorithmically determine and display the orbits for given values of $p$ and $k$. 

1
Table 1.1: Table of Generalized Catalan Orbits

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>27</td>
<td>266</td>
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<td>9</td>
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<td>1</td>
<td>5</td>
<td>33</td>
<td>385</td>
<td>5359</td>
</tr>
</tbody>
</table>

The table is aligned so that values of \( p \) run along the rows, while values of \( k \) run across the columns. We will primarily consider the sequences generated in a single column of the table. These sequences will be analyzed, and formulas will be developed to describe the given column. Using this approach we hope to develop theory for a general trend in the table as we work across the columns. The following theorems will be proven in this paper.
**Theorem 1.1.** For $k = 1$ and $k = 2$ the number of distinct orbits, $\zeta$, is equal to one for all values of $p$. That is,

$$\zeta = 1$$
for all values of $p$.

**Theorem 1.2.** For $k = 3$ the number of distinct orbits, $\zeta$, for a given $p$ value is

$$\zeta = \frac{p}{2} \quad \text{if } p \text{ is even}$$
$$\zeta = \frac{p + 1}{2} \quad \text{if } p \text{ is odd}.$$

**Theorem 1.3.** For $k = 4$ the number of distinct orbits, $\zeta$, for a given $p$ value is

$$\zeta = \frac{1}{3}[p^2 + 2p] \quad \text{if } p \equiv 0, 1, 3, 4 \pmod{6}$$
$$\zeta = \frac{1}{3}[p^2 + 2p + 1] \quad \text{if } p \equiv 2, 5 \pmod{6}.$$

**Theorem 1.4.** For $k = 5$ the number of distinct orbits, $\zeta$, for a given $p$ value is

$$\zeta = \frac{1}{48}[25p^3 - 4p] \quad \text{if } p \text{ is even}$$
$$\zeta = \frac{1}{48}[25p^3 + 29p - 6] \quad \text{if } p \equiv 1 \pmod{4}$$
$$\zeta = \frac{1}{48}[25p^3 + 29p + 6] \quad \text{if } p \equiv 3 \pmod{4}.$$
Theorem 1.5. For $k = 6$ the number of distinct orbits, $\zeta$, for a given $p$ value is

\[
\zeta = \frac{1}{10} \left[ 9p^4 - 9p^3 + 14p^2 - 4p \right] \quad \text{if } p \equiv 0, 1, 2, 5, 6, 7, 8 \pmod{10}
\]

\[
\zeta = \frac{1}{10} \left[ 9p^4 - 9p^3 + 14p^2 - 4p + 4 \right] \quad \text{if } p \equiv 4 \pmod{10}
\]

\[
\zeta = \frac{1}{20} \left[ 18p^4 - 18p^3 + 28p^2 - 8p + 8 \right] \quad \text{if } p \equiv 9 \pmod{10}.
\]

In order to approach our goal, we must first develop the basic properties of the Catalan numbers. We begin by considering the regular Catalan numbers and their properties before making the adaptation to generalized Catalan numbers.
2.1 The Basics of Catalan Numbers

The Catalan numbers were named after the Belgian mathematician Eugene Charles Catalan who discovered the connection between the sequence of natural numbers and parenthesized expressions in the 19th century. Prior to Catalan, Leonhard Euler described the sequence while exploring the number of ways to triangulate a polygon. Euler was the first recorded mathematician to encounter the sequence of Catalan numbers [1]. It is interesting that the Catalan numbers were independently discovered by two mathematicians researching different topics in combinatorics. In fact, the Catalan numbers occur frequently in more than one hundred combinatorial settings [2]. We begin our research of Catalan numbers by examining some of the most common realizations of the sequence of Catalan numbers.

The first realization that we will consider is the scenario that led to Euler’s discovery of the Catalan numbers. We wish to enumerate the number of ways in which a given regular polygon can be triangulated. We are given a regular polygon with \((n + 2)\) sides and wish to divide the interior into triangles with nonintersecting diagonals. The number of ways in which this can be accomplished is referred to as
the $n$th Catalan number, $C_n$ [3]. By convention, $C_0 = 1$. Below we illustrate the
triangulations for $n = 0, 1, 2, 3, \text{ and } 4$.

It should be noted that in each case, exactly $n$ triangles are formed using the
interior of the regular polygon. We also note that many of the arrangements share
the same general structure of diagonals. This fact will become important in later
sections of this paper. From the illustration in Figure 2.1, we see that a sequence $C_n =
\{1, 1, 2, 5, 14, \ldots \}$ is formed. The following formula can be developed to determine the
$n$th Catalan number given a particular value of $n$. 

Figure 2.1: Triangulations for $n = 0$ Through $n = 4$
Although the argument used in the proof of this result is not difficult, we will not show the proof here. Instead we will consider another common realization, the strictly binary trees. A strictly binary tree is a tree structure where each node has either two child nodes or zero child nodes. The $n$th Catalan number also represents the number of strictly binary trees with $n$ interior nodes. The figure below illustrates the different binary tree configurations for $n = 0$ through $n = 3$.

![Figure 2.2: Strictly Binary Trees for $n = 0$ Through $n = 3$](image)

We will show that the trees pictured above also generate the sequence of numbers described by $C_n = \frac{1}{n+1} \binom{2n}{n}$. The following lemma assumes a particular
condition on a sequence $C_n$ is met and proves that under these conditions $C_n = \frac{1}{n+1} \binom{2n}{n}$. The proof of Lemma 2.1 is adapted from the information presented in [3].

**Lemma 2.1.** Let $C_n$ be a sequence of integers satisfying the recurrence relation defined by $C_0 = 1$ and $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$. Then

$$C_n = \frac{1}{n + 1} \binom{2n}{n}.$$ 

**Proof.** Suppose $C_0 = 1$ and $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$. Let $g(x)$ be the generating function for $C_n$, so that $g(x) = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n + ...$. Then,

$$(g(x))^2 = C_0^2 + (C_0 C_1 + C_1 C_0) x + ... + (C_n C_0 + C_{n-1} C_1 + ... + C_0 C_n) x^n + ...$$

$$= C_1 + C_2 x + C_3 x^2 + ... + C_{n+1} x^n + ....$$

Next we consider $g(x)$ and make the following observations.

$$g(x) - 1 = g(x) - C_0$$

$$= C_1 x + C_2 x^2 + ... + C_n x^n + ...$$

Thus,

$$\frac{g(x) - 1}{x} = C_1 + C_2 x + ... + C_{n+1} x^n + ...$$

$$= (g(x))^2.$$
Thus it can be concluded that \((g(x))^2 = \frac{g(x) - 1}{x}\). By rewriting this equation as \(x(g(x))^2 - g(x) + 1 = 0\) and using the quadratic formula, we conclude that the generating function for \(C_n\) can be expressed as

\[
g(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
\]

Let \(f(x) = \sqrt{1 - 4x}\). Using basic calculus and induction on \(n\), we note that

\[
f^{(n)}(x) = -2^n(1)(3)(5)...(2n - 3)(1 - 4x)^\left(\frac{-2n + 1}{2}\right).
\]

From this we can conclude that the Taylor series for \(f(x)\) is

\[
f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + ... \\
= 1 - 2\frac{x}{1!} - 4\frac{x^2}{2!} - ... - 2^n(1)(3)...(2n - 3)\frac{x^n}{n!} - ... .
\]

We then recall that \(g(x) = \frac{1 + f(x)}{2x}\). Since the coefficients of \(g(x)\) must be positive then we require that \(g(x) = \frac{1 - f(x)}{2x}\) so that,

\[
g(x) = \frac{1}{2x} (1 - f(x)) \\
= \frac{1}{2x} \left( 2\frac{x}{1!} + 4\frac{x^2}{2!} + ... + 2^n(1)(3)...(2n - 3)\frac{x^n}{n!} + ... \right) \\
= 1 + 2\frac{x}{2!} + ... + 2^n(1)(3)...(2n - 1)\frac{x^n}{(n + 1)!} + ... .
\]
We now claim that the coefficient of $x^n$ is equal to $C_n = \frac{1}{n+1} \binom{2n}{n}$. We show this by expanding $\frac{1}{n+1} \binom{2n}{n}$ to obtain the desired coefficient.

\[
\frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{(n!(n!)} = \frac{1}{(n+1)!} \frac{(2n)!}{n!} = \frac{1}{(n+1)!} \left[ \frac{(2n)(2n-2)(2n-4)...(4)(2)(2n-1)(2n-3)...(3)(1)}{n!} \right] = \frac{1}{(n+1)!} \left[ \frac{n!2^n(2n-1)(2n-3)...(3)(1)}{n!} \right] = \frac{2^n(1)(3)...(2n-1)}{(n+1)!}
\]

Thus, we have proven that if the recurrence relation is satisfied then it follows that the coefficient of $x^n$ in the generating function is exactly the $n$th Catalan number, $C_n$. \qed

Lemma 2.1 enables us to easily prove that a given combinatorial object is counted by the Catalan numbers by simply proving that the count of the given combinatorial object satisfies the recurrence relation. We now prove that the number of triangulations of a regular $(n+2)$-gon and the number of binary trees with $n$ interior nodes is exactly $C_n$.

**Theorem 2.1.** The number of triangulations of a regular $(n+2)$-gon formed using nonintersecting diagonals is exactly $C_n = \frac{1}{n+1} \binom{2n}{n}$.

**Proof.** We note that in the case of $n = 0$ the regular $(n+2)$-gon that is considered is a line segment. By convention, we consider the number of triangulations of a line
segment to be equal to one, since a single collapsed triangle forms a line segment. Thus $C_0 = 1$ is satisfied. For the case of $n = 1$ the regular polygon under consideration is a triangle. Since the original shape is a triangle there is exactly one way in which to triangulate the shape. Then, $C_1 = C_{0+1} = \sum_{k=0}^{0} C_k C_{n-k} = C_0 C_0 = 1(1) = 1$.

Thus the recurrence relation is satisfied for $n = 1$. Now we need to show that the recurrence relation $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ holds for all $n \geq 2$.

Let $n \geq 2$ and consider a regular polygon, $R$, with $(n + 3) \geq 5$ sides. We note that the triangulation of a polygon with $(n + 3)$ sides corresponds to $C_{n+1}$. We arbitrarily select one side of $R$ and refer to it as the base of $R$. In each triangulation of $R$, the base is a side of a particular triangular region, $T$, that is formed. The triangular region containing the base divides the remainder of $R$ into two distinct polygons: $R_1$ with $k + 1$ sides of the original polygon and one side formed by the added diagonal, and $R_2$ with $n-k+1$ sides of the original polygon and one side formed by the diagonal, where $k \in \{1, 2, ..., n-1\}$. Each of the two remaining regions, $R_1$ and $R_2$, is divided into triangular regions using nonintersecting diagonals. Thus the number of ways to triangulate $R_1$ and $R_2$ can be expressed as $C_k$ and $C_{n-k}$ respectively. Thus we can calculate the total number of ways to triangulate $R$, obtaining $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ for all $n \geq 2$. Thus the condition of Lemma 2.1 is satisfied, and we conclude that the number of triangulations of a regular $(n + 2)$ - gon formed using nonintersecting diagonals is exactly the $n$th Catalan number, $C_n$. \qed
Theorem 2.2. The number of binary trees with $n$ interior nodes is exactly $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. In the case of $n = 0$ the binary tree under consideration has zero interior nodes. By convention, there exists only one empty binary tree. Thus, $C_0 = 1$ is satisfied. For the case of $n = 1$ there exists a single interior node. Since there is exactly one node, there is only one way in which to build the binary tree. Then $C_1 = C_{0+1} = \sum_{k=0}^{0} C_k C_{n-k} = C_0 C_0 = 1(1) = 1$. Thus the recurrence relation holds for $n = 1$. We now desire to show that the recurrence relation $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ holds for all $n \geq 2$.

To begin we consider a binary tree, $B$, with $n+1$ interior nodes. We select and remove the root node of the binary tree, dividing the remaining tree structure into two trees, $B_1$ and $B_2$. We take $B_1$ to be the set of $k$ interior nodes that originated from the
left branch of the root node. Similarly, $B_2$ is taken to be the set of $n - k$ interior nodes that originated from the right branch of the root node, where here $k = 0, 1, 2, \ldots, n$. Then we consider the number of binary trees that can be constructed using the $k$ nodes of $B_1$ and the number of binary trees that can be constructed using the $n - k$ nodes of $B_2$. We see that these enumerations result in $C_k$ and $C_{n-k}$ respectively. Thus we can calculate the total number of binary trees with $n + 1$ interior nodes, obtaining $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ for all $n \geq 2$. Thus the condition of Lemma 2.1 is satisfied, and we conclude that the number of binary trees with $n$ interior nodes is exactly the $n$th Catalan number, $C_n$.

2.2 The Basics of Generalized Catalan Numbers

The generalized Catalan numbers rely on two indices, $p$ and $k$, instead of the single value of $n$ used in the regular case. The increase in indices is due to the added parameter that is considered in the generalized Catalan numbers. The majority of the realizations of the regular Catalan numbers can easily be expanded to be considered under the generalized Catalan numbers. A simple adaptation is that of regular polygons. Previously we considered the triangulations of a given polygon. The generalized Catalan number will consider the dissections of a given regular polygon into $(p+1)$-gons. The generalized Catalan numbers involve both the size of the given polygon and the size of the sub-polygons which will be created with nonintersecting diagonals. More specifically, the generalized Catalan number, $C_{p,k}$ counts the number of ways to dissect a regular polygon with $((p - 1)k + 2)$ sides into $k$ smaller $(p + 1)$-
gons using exactly \( k - 1 \) nonintersecting diagonals [4]. Because of the increased complexity there is not a single sequence that represents the generalized Catalan numbers. Instead a grid of values is generated. A single sequence of values can be considered if a chosen \( p \) (or \( k \)) is held constant. The formula for the generalized Catalan numbers is known and can be proven using a variety of methods, including that of the dissection of regular polygons. A detailed proof can be found in [4]. The formula is as follows:

\[
C_{p,k} = \frac{1}{(p - 1)k + 1} \binom{pk}{k}
\]

It can be noted that in the above formula, choosing \( p = 2 \) yields the regular Catalan numbers. We now illustrate the concept of the generalized Catalan numbers using the above described notion of the dissection of regular polygons. We consider the case of \( p = 3 \) and \( k = 2 \). We notice that the given polygon has \((3 - 1)2 + 2 = 6\) sides. We wish to create exactly \( k = 2 \) quadrilaterals (since \( p = 3 \) and thus \( p + 1 = 4 \)). We conclude using the above formula that there are exactly \( \frac{1}{5}\binom{6}{2} = 3 \) ways in which this can be accomplished; thus, \( C_{3,2} = 3 \). Figure 2.4 illustrates the three possibilities.

We now consider the more complex case of \( p = 3, k = 3 \). In this case we are given a polygon with \((3 - 1)3 + 2 = 8\) sides which must be divided into three quadrilaterals. We note that this will be accomplished using \((k - 1) = 2\) nonintersecting diagonals. The number of possibilities, illustrated below, is equal to \( \frac{1}{4}\binom{9}{3} = 12 \). By observing Figure 2.5, it can be noted that several diagonalizations contain the
same general diagonal structure. These similarities will be defined and studied in the following section.

2.3 Terminology and Definitions

We have previously noted that there exist many similarities between the diagonalizations (equivalently, polygonizations) which comprise a particular generalized Catalan number, $C_{p,k}$. In fact, in many cases there exist different polygonizations which share the same basic diagonal structure. Dissected polygons which share the same underlying diagonal structure belong to the same orbit.

**Definition.** *Given two polygonizations $P$ and $Q$, we say that $P$ and $Q$ are in the same orbit if and only if applying an element from the dihedral group $D_{2n}$ transforms one polygonization to the other, where $n$ is the number of sides of the original polygon.*

A *generalized Catalan orbit* is an orbit of the underlying geometric object which is counted by the generalized Catalan numbers. As an example, we observe
that there are exactly two distinct diagonalizations which divide an octagon into quadrilaterals which was illustrated in Figure 2.5; thus, there are exactly two generalized Catalan orbits for the values \( p = 3, k = 3 \). We depict these orbits in Figure 2.6.

Our goal is to count the number of generalized Catalan orbits, \( C_{p,k} \). We will approach this goal by considering the columns of Table 1. By considering the columns we will fix the value of \( k \) and allow the value of \( p \) to vary. We will approach our goal
using a variety of techniques. In order to successfully execute these techniques we will need to reference a variety of terminology. Here we define some of the common vocabulary that will be used throughout our analysis.

When examining the orbits using the realization of polygons we will often refer to the *original polygon*. The original polygon is the polygon which is to be dissected. This polygon has exactly \((p - 1)k + 2\) sides. The original polygon will be divided into *subpolygons* with \(p + 1\) sides each. A *key section* is a subpolygon that is formed using one diagonal and exactly \(p\) sides of the original polygon. The polygon below (an orbit of \(p = 2, k = 4\)) has exactly three key sections.

![Figure 2.7: An Orbit with Three Key Sections](image)

Additionally, we will consider the number of orbits under the realization of *p-ary trees*. An *p-ary tree* is a tree where each node has degree \((p + 1)\) or degree one. Nodes of the tree with degree \((p + 1)\) are referred to as the *interior nodes* of the tree, while nodes with degree one are referred to as the *exterior nodes*. The set of interior nodes (and edges which connect them) form the *core* of the tree structure. Below we
illustrate a typical tree denoting the interior nodes with filled circles and the exterior nodes with open circles.

![Figure 2.8: An Illustration of a Tree with Interior and Exterior Nodes](image)

In our analysis we will often consider only the core of the tree structure. When we illustrate the core we will denote key sections with square nodes instead of circular nodes, referring to the nodes as *key nodes*. The core of the tree in Figure 2.8 is depicted below.

![Figure 2.9: An Illustration of the Core of a Tree](image)

This reference of terminology is not exhaustive; we will often define new terminology as needed throughout our analysis. Before beginning the analysis of the number of orbits of the generalized Catalan numbers we will examine a program which was developed to algorithmically count and display the orbits for given values of $p$ and $k$. 

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CHAPTER III
COMPUTATIONAL RESULTS

It is often useful to have a second source with which to verify our results. For this reason we chose to develop a computer program which executes an algorithm to find the distinct orbits for given values of \( p \) and \( k \). The program is not an explicit part of our research; it is merely a tool to be used throughout our analysis of the generalized Catalan orbits. We will now examine the program and the algorithm on which it runs. The development of this program was assisted by Christoper Hirgelt, graduate of The Ohio State University.

The program uses the interface which is illustrated in Figure 3.1. The user enters values of \( p \) and \( k \) into the appropriate boxes in the display and clicks the ‘Calculate’ button. To the right of the button is a display area for information regarding the inputted values. Results for the number of sides of the original polygon, the corresponding generalized Catalan number, the number and size of the sub-polygons, and the number of diagonals used are displayed when the program concludes. The center of the interface is a display area for the original polygon. Below the main display area is the display strip (with a scroll bar as necessary) where the distinct orbits will be depicted. We illustrate the results of the program using the input values of \( p = 3 \) and \( k = 3 \).
Figure 3.1: Program Interface
Figure 3.2: Program Results for Input Values $p = 3$ and $k = 3$
The program first calculates, stores, and displays the information found in the top right corner of the interface. Using the formula \((p - 1)k + 2\), the program is able to determine the number of sides of the original polygon. Additionally it determines the generalized Catalan number, namely \(C_{p,k} = \frac{1}{(p-1)k+1} \binom{pk}{k}\), the details of the sub-polygons \(k\) sub-polygons with \(p+1\) sides each), as well as the number of diagonals \((k - 1)\).

Next the computer draws the original polygon. The first point is plotted at the top center of the display area. Using the stored number of sides of the original polygon the computer determines the placement of the remaining vertices of the polygon using angles. Once the points (corresponding to the vertices of the original polygon) are plotted, the connecting edges are drawn. The resulting figure is stored and displayed on the interface.

Once the original polygon has been drawn and stored the program is ready to determine the orbits. Since the computer cannot use visuals to determine if two polygonizations are in the same orbit, it must use an extended algorithm to draw all possible polygonizations and then compare the results to eliminate duplicates. The algorithm is carefully constructed to reduce the number of calculations that the computer must run. Throughout our description of the main algorithm we will illustrate concepts using the values of \(p = 3\) and \(k = 4\).

The algorithm begins by drawing the first key section. The key section is drawn by connecting the top center node (or top center left node if two nodes are centered at the top of the display) with the node that is \(p\) nodes in the clockwise
direction. The position of this drawn diagonal will not move throughout the course of the algorithm. If we were to construct the first key section in each possible location on the original polygon then an increased number of duplicates would be generated throughout the course of the algorithm. By fixing the first diagonal in this arbitrary manner we greatly reduce the number of calculations of the program. The program now begins to place the remaining \( k - 2 \) diagonals.

![Figure 3.3: Placement of the First Key Section](image)

The second diagonal is drawn by placing the right endpoint of the diagonal on the node where the first key section terminated (the right endpoint of the first key section). The left endpoint of the diagonal is determined so that a sub-polygon with \( p + 1 \) sides is formed between the first diagonal and the second diagonal. The computer will now consider all possible arrangements of the remaining \( k - 3 \) diagonals using this particular position of the second diagonal before it returns to examine alternative positions of the second diagonal.

The placement of the third diagonal will begin in the same way. The right endpoint of the third diagonal will share the same node as the right endpoint of the
second diagonal. The left endpoint will then be determined so that a sub-polygon with \( p + 1 \) sides is formed. If more than three diagonals are to be used then the algorithm will continue to the fourth diagonal before returning to consider alternative placements of the third diagonal. We will discuss the algorithm in terms of three diagonals; however, it can easily be extended for cases with greater numbers of diagonals. When the algorithm reaches the last diagonal (in this case, the third diagonal) it will begin to consider alternative placements of the diagonals. Once the first placement of the third diagonal is drawn the computer will consider the second placement by placing the right endpoint of the third diagonal on the node which is one position past the previous starting point in the clockwise direction. The computer will continue in this fashion, moving the third diagonal by one node until all nodes have been considered. Each arrangement is stored for later comparison.

In Figure 3.5 the starting place of the third diagonal is labeled by the letter \( s \). We can see that already we have begun to generate duplicate polygonizations. It is difficult to compute all possible orbits without producing duplicate polygonizations or
polygonizations which belong to the same orbit. Although the program will effectively find all possibilities and remove all duplicates it is possible that the efficiency of the algorithm could be improved.

Now that all possibilities of the third diagonal have been considered (using the first placement of the second diagonal) we return to consider another possibility of the second diagonal. We choose the next placement in the same way as the third diagonal, by moving the diagonal one node in the clockwise direction. Once the diagonal is placed we must again consider all possible placements of the third diagonal. This recursive action continues until we have considered all possible combinations of the second and third diagonals (recall that the position of the first diagonal is fixed).
Figure 3.6 and Figure 3.7 illustrate the recursive process for polygonizations with three and four diagonals respectively.

Throughout the recursive process the computer checks for duplicate polygonizations and polygonizations which belong to the same orbit. For simplicity we will discuss only the final check which the computer executes before the program’s conclusion. Once the computer has examined all possible polygonizations using the recursive algorithm it must ensure that only distinct orbits will be printed to the screen. To do this the computer compares the degree sequences of the polygonizations which have been generated. The degree sequence of a polygonization is a sequence of numbers which describes (in order) the number of edges which connect to the vertices of the polygon. The degree sequences are compared in multiple ways to ensure the removal
Figure 3.7: Algorithm to Compute Possible Orbits with Four Diagonals

of all duplications. Sequences are considered at each possible starting point on the polygon in both the clockwise and counterclockwise directions. The program loops continually in order to compare each pair of polygonizations in this manner. Once all duplicates have been removed the program finishes by printing the distinct orbits to the display area on the interface. We illustrate the program results of $p = 3$ and $k = 4$ in Figure 3.8.
Figure 3.8: Program Results for Input Values $p = 3$ and $k = 4$
CHAPTER IV

MAIN RESULTS

4.1 Technique

To develop the main results of this thesis, we consider the columns of Table 1. It can be noted that the rows of Table 1 can also be considered, as is done in [5] for the row $p = 2$. By considering columns, we are examining the number of orbits of a pair $(p, k)$ where $k$ is fixed and $p$ is allowed to vary. In our most common realization this approach corresponds to fixing the number of sub-polygons formed and varying the size of both the sub-polygons and the original polygon. We wish to develop a closed formula for the columns with values $k = 1$ through $k = 6$. To accomplish this we will utilize many techniques of combinatorics and group theory. The analysis of each column will begin by using the basic definitions associated with the generalized Catalan numbers. We will use the fact that the given polygon with $(p - 1)k + 2$ sides will be dissected into exactly $k$ polygons with $(p + 1)$ sides using $k - 1$ diagonals. This will allow us to begin to systematically count the number of orbits. We will produce closed formulas for the first six columns of the table.

Throughout our analysis we will utilize the realization of $p$-ary trees. In order to count the number of orbits of each tree structure, some notation will be useful.
We define the number of partitions of $k$ having length $l$ as $P_{[l][k]}$, and the number of compositions of $k$ having length $l$ as $C_{[l][k]}$. Recall that a partition is a way of writing a given number as a sum of positive integers where the order of the summands does not matter. In the case of a composition the order of the summands is also considered. We will now state and prove several lemmas concerning the above notation which will prove useful in creating a formula for the number of orbits that is expressed only in terms of $p$. Additionally, we will state an important lemma which will be used in several instances throughout the proofs in this chapter.

**Lemma 4.1.** The number of partitions of $(p - 1)$ of length two is exactly $\lceil \frac{p - 1}{2} \rceil$.

Considering the parity of $p$, we can conclude that

\[
P_{[2][p-1]} = \begin{cases} 
\frac{p}{2} & \text{if } p \text{ is even} \\
\frac{p + 1}{2} & \text{if } p \text{ is odd}
\end{cases}
\]

**Proof.** We wish to determine the number of partitions of $p - 1$ of length two, equivalently, the number of ways to select $a$ and $b$ so that $a + b = p - 1$ where the order of $a$ and $b$ is irrelevant. We begin by selecting $a$. We may choose this number from the values zero through $p - 1$; this yields a total of $p$ possibilities. Since $a + b = p - 1$, then the value of $b$ is completely determined by the selection of $a$. We divide the number of possibilities by two since $a + b = b + a$, careful to exclude the single case
when \( p \) is odd and \( a = b \). This case separates our answer based on the parity of \( p \), yielding the desired result.

\[\square\]

**Lemma 4.2.** The number of compositions of \( k \) of length two is equal to \( k + 1 \). That is, \( C_{[2][k]} = k + 1 \).

**Proof.** Let \((a, b)\) be an arbitrary composition of \( k \) of length two for \( 0 \leq a, b \leq k \). By the definition of a composition, \( a + b = k \). Since \( k \) is known, the assignment of \( a \) uniquely determines the value of \( b \). Since \( 0 \leq a \leq k \), there are \( k + 1 \) choices for \( a \). Thus there are \( k + 1 \) compositions of \( k \) of length two, and \( C_{[2][k]} = k + 1 \).

\[\square\]

**Lemma 4.3.** The number of compositions of \( p - 3 \) of length four is equal to \( \frac{1}{6}(p^3 - 3p^2 + 2p) \). That is, \( C_{[4][p-3]} = \frac{1}{6}(p^3 - 3p^2 + 2p) \).

**Proof.** The number of compositions of \( p - 3 \) of length four can be described as the number of ways to arrange \((a, b, c, d)\) such that \( a + b + c + d = (p - 3) \) and \( 0 \leq a, b, c, d \leq p - 3 \). This is precisely the number of \((p - 3)\)-combinations of the multiset \( M = \{\infty \cdot 1, \infty \cdot 2, \infty \cdot 3, \infty \cdot 4\} \) which is equal to \( \binom{(p-3)+(4-1)}{4-1} = \binom{p}{3} = \frac{p(p-1)(p-2)}{6} = \frac{1}{6}(p^3 - 3p^2 + 2p) \) [3].

\[\square\]

**Lemma 4.4.** (Burnside’s Lemma) For a finite group \( G \), acting on a set \( X \), the number of orbits of \( X \) is equal to \( \frac{1}{|G|} \sum_{g \in G} \chi(g) \) where \( \chi(g) \) represents the number of elements in \( X \) that are fixed by \( g \).
4.2 The Columns $k = 1$ and $k = 2$

Once the information given by the definition of the generalized Catalan numbers is analyzed it is simple to see that the cases of $k = 1$ and $k = 2$ are trivial.

**Theorem 4.1.** For $k = 1$ and $k = 2$ the number of distinct orbits, $\zeta$, is equal to one for all values of $p$. That is,

$$\zeta = 1 \quad \text{for all values of } p.$$ 

4.2.1 The Column $k = 1$

We begin our analysis of the column corresponding to $k = 1$ by considering the polygon which is to be dissected. By the definition of the generalized Catalan numbers this polygon has exactly $(p - 1)k + 2$ sides. When $k = 1$, we find that the polygon has exactly $p + 1$ sides. Our goal is to dissect this polygon into exactly one $(p + 1)$-gon. Since our given polygon has $(p + 1)$ sides, there is exactly one way for the given polygon to be constructed completely of $(p + 1)$-gons. Thus, the column $k = 1$ is trivial, with a value of one for all values of $p$.

4.2.2 The Column $k = 2$

In the column where $k = 2$, the given polygon is to be dissected into two sub-polygons using exactly one diagonal. To determine the number of orbits for a particular value of $p$ when $k = 2$ we examine the group which acts upon the polygon dissection. We will show that the number of orbits is equal to one for all values of $p$. If we arbitrarily
select two polygonizations we are able to show they are in the same orbit. We do so by applying a series of group elements to the second polygonization to yield a polygonization identical to the first. We illustrate with a simple example.

![Figure 4.1: An Example Using $k = 2$ and $p = 4$](image)

By simply rotating the polygon in Figure 4.1 we are able to recover the same configuration. Since one polygonization can be made identical to the other by using a member of the dihedral group, the two elements are in the same orbit. Since the diagonal must always be drawn such that $p$ polygon sides are arranged on either side of the diagonal, all configurations with one diagonal are in the same orbit. Thus, the column $k = 2$ has a value of one for any given value of $p$. 

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4.3 The Column $k = 3$

The $k = 3$ column examines polygons with $(p - 1)3 + 2 = 3p - 1$ sides. We wish to dissect the given polygon into three polygons with $p + 1$ sides each by using two diagonals. Using this information we will construct the following formula for the number of orbits in terms of a given value of $p$.

**Theorem 4.2.** For $k = 3$ the number of distinct orbits, $\zeta$, for a given $p$ value is

\[ \zeta = \begin{cases} 
\frac{p}{2} & \text{if } p \text{ is even} \\
\frac{p + 1}{2} & \text{if } p \text{ is odd.}
\end{cases} \]

**Proof.** In order to count the number of orbits we begin by arbitrarily placing the first diagonal. Once the first diagonal is placed we must determine the number of orbits that are formed when we place the second diagonal. By ignoring the key section that is formed by the first diagonal, we see that we are left with $3p - 1 - (p - 1) = 2p$ sides. The subtracted term comes from the fact that we are removing $p$ sides of the original polygon and also adding an additional side from the diagonal.

We begin by considering the case where $p$ is even. Since there are $2p$ sides, there are $2p$ places (vertices) from which we may begin the second diagonal. It can be seen in the example below that the selection of a starting place eliminates a total of four places at which the second diagonal can be placed. We arrive at four by considering the place where we have started to be removed as well as the place in which the diagonal ends (predetermined by the requirement of a subpolygon with $p + 1$
sides), since starting at this point would draw the same diagonal. Additionally we must consider the effect of the symmetry of the polygon. The symmetry is illustrated in the example, by the dotted line cutting directly through the key section and the remaining sides of the original polygon. The two end points of the diagonal are removed from consideration as well as the vertices which correspond to these endpoints under the line of symmetry (since the diagonal created by the endpoints under the symmetry would create a diagonalization that belongs to the same orbit). Since each unique starting point must remove four possible points we can conclude the number of possible starting points is \( \frac{2p}{4} = \frac{p}{2} \) for even values of \( p \).

![Figure 4.2: An Example of \( k = 3 \) for Even \( p = 4 \)](image)

Next we consider the case where \( p \) is odd. This case is very similar to the previous case with one exception. Every possible starting point eliminates exactly four points with the exception of one particular diagonal. The diagonal which is parallel to the first key section only removes two points since it is bisected by the symmetry. This is illustrated in the example below. In order to determine the
number of orbits, we subtract the two points which are removed by this case from $2p$ (effectively removing this configuration) and then divide by four as before. We then add one to the result to include the parallel diagonalization which we excluded. The number of orbits is then $\frac{2p-2}{4} + 1 = \frac{p-1}{2} + 1 = \frac{p+1}{2}$ for odd values of $p$.

![Diagram of orbits](image)

Figure 4.3: An Example of $k = 3$ for Odd $p = 5$

We summarize our results for the column $k = 3$ below.

\[
\zeta = \begin{cases} 
\frac{p}{2} & \text{if } p \text{ is even} \\
\frac{p+1}{2} & \text{if } p \text{ is odd}
\end{cases}
\]

Additionally, one can easily develop the generating function for the $k = 3$ column, shown below.

\[
h(x) = \frac{1}{(1 - x)^2(1 + x)} = 1 + x + 2x^2 + 2x^3 + 3x^4 + ...
\]
4.4 The Column $k = 4$

4.4.1 Polygon Approach

For the case of the $k = 4$ column, exactly $(k - 1) = 3$ diagonals are used to divide a given polygon with $(p - 1)4 + 2$ sides. To count the number of orbits, we consider two cases: the collection of diagonal configurations with two key sections and the collection of diagonal configurations with three key sections. Recall that a key section is a sub-polygon that is formed when a diagonal forms a sub-polygon that includes exactly $p$ exterior sides of the polygon. It should be noted that these cases partition the orbit configurations of the $k = 4$ column. Thus, the total number of orbits can be determined by summing the number of orbits with two key sections and the number of orbits with three key sections.

We first examine orbits with three key sections. The given polygon has $(p - 1)4 + 2 = 4p - 2$ sides which must be divided to form four smaller $(p + 1)$-gons. By considering that each key section removes $p$ exterior sides of the original polygon, we can see that $(4p - 2) - 3p = (p - 2)$ sides remain. Figure 4.4 illustrates this configuration.

Using the labels from Figure 4.4, it can be seen that $a + b + c = p - 2$ where $a, b$ and $c$ are nonnegative integers that count the number of exterior sides in each marked gap of the polygon. In order to avoid overcounting the number of orbits, we use a partition to determine the number of ways to assign values to $a, b$ and $c$. The placement of $a, b$ and $c$ in the polygon form the shape of a triangle. Since the
triangular shape has many symmetries we use a partition to fix the values of $a$, $b$ and $c$ in relation to the triangular structure. By fixing these positions we effectively eliminate the possibility of counting polygonizations which belong to the same orbit. Thus, the number of ways to arrange the diagonals is equal to the number of partitions of $(p - 2)$ of length at most three.

We can develop the following generating function to describe the number of partitions of $(p - 2)$ with length three or less. By conjugating the partition, we see that counting the partitions of $(p-2)$ having no more than three parts is equivalent to counting the partitions of $(p-2)$ with no part bigger than three. Thus our generating function has three parts: \( \frac{1}{1-x} \) counts the number of rows in the Young diagram of the partition of length one, \( \frac{1}{1-x^2} \) counts the number of rows of length two, and \( \frac{1}{1-x^3} \) counts the number of rows of length three. By multiplying these generating functions, we develop a function that counts the number of rows of length three or less. Since we are considering partitions of $p - 2$ we must multiply by a factor of $x^2$. 

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Thus $g_3(x)$ is the generating function for the number of orbits with three key sections.

Next we consider the case where the polygon has exactly two key sections. In this case, the third diagonal must occur between the two key sections. Thus, we consider the possible gaps, denoted $g$, between the two key sections. The remaining diagonal must begin at one of the points in the gap (including the endpoints where the diagonal forming the key section meets the polygon). To account for symmetry and avoid overcounting, we will consider the first $\lceil \frac{g+1}{2} \rceil$ points in the gap. It is clear that a unique diagonal can be formed at each point in this selection of the gap points. We illustrate a small example below.

Figure 4.5: Example of Two Key Section Process Using $p = 3$ where $g = 2$
Note that in the first row of the illustration we began the third diagonal at a gap of zero in each picture; however, by applying a group element to the polygon, we notice that the diagonal structures are exactly the same, meaning they belong to the same orbit. For each gap that we consider, it will be the case that examining a particular gap from one side of the polygon will be equivalent to examining the same gap from the opposite side of the polygon. For values of $g$ greater than or equal to zero, $\left\lceil \frac{g+1}{2} \right\rceil$ corresponds to the values of the $k = 3$ column in Table 1. Thus summing over the unique orbits from the possible gap sizes is equivalent to summing down the $k = 3$ column from $p = 1$ to the current $p$ value being evaluated (this suggests a certain recursive structure in the table that merits further study). To determine the generating function for this scenario we will use the fact that multiplication by $\frac{1}{1-x}$ sums the coefficients of a given generating function when multiplied with the given generating function. We use the generating function for the $k = 3$ column and this fact to develop the generating function for the case of two key sections. We must additionally multiply by a factor of $x$ in order to correctly sum the terms of $g_2$ and $g_3$.

$$g_2(x) = \frac{x}{(1-x)^3(1+x)}$$

To develop our final result for the $k = 4$ column we sum the results from the three key section case and the two key section case. This will provide us with a generating function for the entire column. The generating function is determined and simplified below.
\[ g(x) = \frac{x}{(1-x)^3(1+x)} + \frac{x^2}{(1-x)(1-x^2)(1-x^3)} \]

\[ = \frac{x^3 + 2x^2 + x}{(1-x)(1-x^2)(1-x^3)} \]

\[ = x + 3x^2 + 5x^3 + 8x^4 + 12x^5 + 16x^6 + 21x^7 + 27x^8 + 33x^9 + \ldots \]

We can see that the coefficients in the generating function correspond to the values in the \( k = 4 \) column of the table.

4.4.2 Relating Polygons and Tree Structures

While we were able to develop a generating function for the number of orbits for \( k = 4 \) and a given value of \( p \), we were so far unable to construct a closed formula for the number of orbits when \( k = 4 \). Because of this deficiency, we examine other approaches to counting the number of orbits for given values of \( p \) and \( k \). We have previously demonstrated that both regular polygons and binary trees give rise to representations of the regular Catalan numbers. Since most realizations of the classical Catalan numbers can be extended to generalized Catalan numbers, it is natural that the generalized Catalan numbers can be represented by both polygons and tree structures. In fact, in 1991 Peter Hilton and Jean Pedersen established a bijection between the dissections of a given polygon with \((p - 1)k + 2\) sides into \( k \) polygons with \( (p + 1) \) sides and the \( p \)-ary trees with \( k \) source nodes [4]. We will not reproduce this proof;
instead, we will demonstrate the method used to transition between a polygon and a tree structure.

To make this transition we begin by placing a single node on the exterior of each side of the original polygon. These will be referred to as the exterior nodes of the tree. Next we place a node in each interior subpolygon of the original polygon. These will be called the interior (or source) nodes. Next we consider the edges of the tree. Edges are added to the tree structure so that for each subpolygon, exactly one edge intersects each side of the subpolygon. An edge that intersects the side of the subpolygon connects with either the node of the adjacent subpolygon or the node corresponding the side (which was intersected) of the original polygon. Note that there are exactly $k$ interior nodes, each of which has exactly $p + 1$ connecting edges. Additionally, there are $(p - 1)k + 2$ exterior nodes, each of which has exactly one connecting edge. The core of a tree is defined to be the interior nodes of the tree and the edges that connect them. Equivalently, the core can be formed by removing all nodes of degree one from the original tree structure. We observe that a key section is uniquely formed by an interior node that is connected to exactly $p$ adjacent exterior nodes. Once the tree has been formed inside of the polygon, we remove the polygon and align the source nodes in the simplest fashion possible. This alignment does not change the underlying structure of the tree; however, it enables us to more readily observe congruences between tree structures. Figure 4.6 illustrates this concept.

Though a bijection exists between the tree structures and polygon dissections of $C_{p,k}$, it remains to describe the behavior of the orbits with respect to this bijection.
In order to determine whether two tree structures are in the same orbit we must develop a way to compare tree structures. We note that two trees may be isomorphic as graphs and yet correspond to different orbits, which we illustrate with an example below.

Figure 4.7: Isomorphic Graphs Corresponding to Different Orbits
Two graphs, $X$ and $Y$, are said to be isomorphic if there exists a bijection $f$ between the set of vertices of $X$ and the set of vertices of $Y$ such that any two vertices $u$ and $v$ of $X$ are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in $Y$. By the previous definition, the two tree structures shown above are isomorphic. However, it is easy to see that the two trees correspond to different orbits. The tree on the right has a line of symmetry along the horizontal axis while the tree on the left does not. This simple observation prevents the two trees from belonging to the same orbit. To circumvent this issue, we determine congruences between tree structures by introducing the concept of the oriented degree sequence.

To determine an oriented degree sequence of a given tree we first select an exterior node of degree one, referred to as $v$. From this node, we travel to the adjacent source node, $w$, and begin to record the degrees of all nodes directly connected to the source node (beginning with the exterior node from which we started) while moving in a predetermined direction (clockwise or counterclockwise). Once the degree of each node adjacent to $w$ has been recorded we move to the next source node, $z$, that is in the direction (clockwise or counterclockwise) we have chosen for our oriented degree sequence. Again, we move around source node $z$ recording the degree values of each connected node starting with the source node which we last examined ($w$). This process is continued until we reach a source node that is connected only to source nodes which have already been counted. We refer to as source node $n$. From $n$ we reverse our steps until we reach a source node that has not been counted, and we record the connected degree values in the same fashion. We continue this process until
we reach the first source node considered so that all source nodes have been examined. Upon completion, the oriented degree sequence will have \( k \) blocks of \( p + 1 \) numbers, each of which corresponds to a degree. Notice a given tree will yield different degree sequences, depending on our choice of starting node and orientation. We will now illustrate this concept with an example using \( k = 5 \) and \( p = 3 \).

The illustration in Figure 4.8 depicts a typical tree structure. To develop an oriented degree sequence we choose the clockwise direction (for instance) and an exterior node from which to start. We note our starting node on the diagram and label the source nodes in the order in which they will be examined. In the process we will choose our next source node by selecting the next connected source node which is in the clockwise direction. We note that when we reach node four there are no connecting source nodes which have not been recorded. Thus in this example, source node \( n \) corresponds to source node four. Thus we will begin the backwards process. From node four we move back to node three and move again in the clockwise direction (starting with source node four). This brings us to source node five which has not yet been recorded. Once this source node is recorded the process is complete. The resulting oriented degree sequence is 1141 4141 4144 4111 4111.
Using the oriented degree sequence we are able to determine relationships between trees which result in the same orbit. The following theorem use the oriented degree sequence to formally state the results which enable us to count the number of orbits of the generalized Catalan numbers by using tree structures. For proofs of the following theorems refer to Joseph Auger’s Master’s Thesis entitled *Orbits of the Dissected Polygons of the Generalized Catalan Numbers* [6].

**Theorem 4.3.** Two trees $A$ and $B$ correspond to the same orbit if and only if:

1. They are isomorphic as graphs, and
2. For some vertex $a \in A$ and some vertex $b \in B$, the oriented degree sequence for $a$ (in some direction) is the same as the oriented degree sequence for $b$ (in some direction).

We illustrate this theorem in Figure 4.9 using a clockwise rotation to formulate the oriented degree sequence in each structure (each with a different starting point). The oriented degree sequence for each tree is listed beneath the tree. It can be seen that the first block of the left tree’s oriented degree sequence can be reordered...
to obtain the last block of the right tree’s oriented degree sequence. The remaining blocks can be paired similarly showing that the assumptions of the theorem are satisfied.

![Diagram of oriented degree sequence](image)

Figure 4.9: An Illustration of the Oriented Degree Sequence in Relation to Orbits

Next we will use these results to consider the $k = 4$ column using tree structures.

4.4.3 Tree-Structure Approach

By utilizing the relationship between binary trees and the polygons which we wish to dissect, we are able to develop a closed formula for the number of orbits when $k = 4$ for any value of $p$. When $k = 4$ we see that the cases we considered under the polygon approach (partitioned according the the number of key sections) correspond directly to the cores of tree structures which we will consider. We note that this will not always be the case. As the values of $p$ and $k$ get larger, there could be many different cores with the same number of key sections.
Figure 4.10 depicts the two core tree structures under consideration. Recall that the core of a tree structure depicts only the interior tree nodes, where fully occupied nodes (centered in a key section) are represented by square nodes.

Theorem 4.4. For \( k = 4 \) the number of distinct orbits, \( \zeta \), for a given \( p \) value is

\[
\zeta = \frac{1}{3}(p^2 + 2p) \quad \text{if } p \equiv 0, 1, 3, 4 \pmod{6}
\]

\[
\zeta = \frac{1}{3}(p^2 + 2p + 1) \quad \text{if } p \equiv 2, 5 \pmod{6}.
\]

Proof. The theorem will be proven by examining the above tree structures. First we will consider the structure with three filled interior nodes. We label the areas surrounding the center node with \( A, B \) and \( C \); these are the only areas eligible to receive additional edges. These variables represent the number of edges which are connected to the center node at the position of the label. Since each node must have exactly \( p+1 \) connections we see that the center node must still receive \((p+1)−3 = p−2\) connections. Thus, it must be true that \( A + B + C = p - 2 \).
Because of what we now know about oriented degree sequences and orbits, it is clear that our goal is to examine the underlying tree structure and count the possible values of $A, B$ and $C$ that yield inequivalent trees. To count the possible values of $A, B$ and $C$, we will consider Burnside’s Lemma (see lemma 4.4). In the application of Burnside’s lemma we will allow the acting group to be the group that consists of all of the symmetries of the core of the tree structure in question. It is easy to see that the underlying group structure of the core is that of $D_6$ since the symmetries of the tree structure are exactly the symmetries of an equilateral triangle. Thus we will consider the elements $1, r_{120}, r_{240}$ and $f_i$ where $f_i$ is the flip that bisects node $i = A, B, C$.

We first consider the number of group elements fixed by the group identity. This can be easily expressed as the number of compositions of $p - 2$ of length three. Thus, $\chi(1) = C_{[3][p-2]}$ where $C_{[3][p-2]}$ can be calculated as a permutation of a multiset, yielding $\frac{1}{2}(p^2 - p)$ for all values of $p$. Next we consider the rotational elements. In order to be fixed by a rotation, we must have $A = B = C$. Thus a rotation fixes at most one element, and $\chi(r_{120}) = \chi(r_{240}) = 1$ if and only if $3|(p - 2)$, and $\chi(r_{120}) = \chi(r_{240}) = 0$ if $3 \nmid (p - 2)$. Finally we consider the flip $f_i$. For simplicity, we compute $\chi(f_C)$ and
note that the values of \( \chi(f_A) \) and \( \chi(f_B) \) can be calculated in a similar manner. The flip \( f_C \) fixes the value of \( C \). Then in order to be fixed under the flip, it must be true that \( A = B \) which is possible only when \( 2|(A + B) \). We note that since \( 0 \leq C \leq p - 2 \), there are exactly \( p - 1 \) different sums for \( A + B \). Half of these sum values will be even, yielding \( \lceil \frac{p - 1}{2} \rceil \) ways for \( A = B \). Thus, \( \chi(f_A) = \chi(f_B) = \chi(f_C) = \lceil \frac{p - 1}{2} \rceil \) for all values of \( p \). We summarize and simplify these results below.

\[
\begin{align*}
\chi(1) &= \frac{1}{2}(p^2 - p) \\
\chi(r_{120}) &= \chi(r_{240}) = 1 \quad \text{if } 3|(p - 2) \\
\chi(r_{120}) &= \chi(r_{240}) = 0 \quad \text{if } 3 \nmid (p - 2) \\
\chi(f_A) = \chi(f_B) &= \chi(f_C) = \frac{p - 1}{2} \quad \text{if } p \text{ is odd} \\
\chi(f_A) &= \chi(f_B) = \chi(f_C) = \frac{p}{2} \quad \text{if } p \text{ is even.}
\end{align*}
\]

By combining the above results using Burnside’s Lemma, we develop a formula for \( \zeta_1 \), the number of orbits of the first tree structure given a particular value of \( p \). By examining the conditions on \( p \), we note that we must create six distinct cases.
Now we consider the second tree structure with two filled nodes and two nodes which can receive additional edges. We label the areas which are eligible to receive connections by $A, B, C$ and $D$ in the below illustration. We observe that it must be true that $A + B = p - 1$ and $C + D = p - 1$.

![Figure 4.12: Second Tree Structure](image)

In order to count the number of orbits, $\zeta_2$, of this tree structure we will again use Burnside’s Lemma. In this case the group acting on the trees is not as obvious. We will, instead of using a typical dihedral group, create our own group consisting of the symmetries of the tree structure. We see that we have four group elements: the

\[
\zeta_1 = \frac{1}{12} (p^2 + 2p) \quad \text{if } p \equiv 0 \pmod{6}
\]
\[
\zeta_1 = \frac{1}{12} (p^2 + 2p - 3) \quad \text{if } p \equiv 1 \pmod{6}
\]
\[
\zeta_1 = \frac{1}{12} (p^2 + 2p + 4) \quad \text{if } p \equiv 2 \pmod{6}
\]
\[
\zeta_1 = \frac{1}{12} (p^2 + 2p - 3) \quad \text{if } p \equiv 3 \pmod{6}
\]
\[
\zeta_1 = \frac{1}{12} (p^2 + 2p) \quad \text{if } p \equiv 4 \pmod{6}
\]
\[
\zeta_1 = \frac{1}{12} (p^2 + 2p + 1) \quad \text{if } p \equiv 5 \pmod{6}
\]
identity, a rotation of 180 degrees ($r_{180}$), a flip through the vertical axis ($f_v$), and a flip through the horizontal axis ($f_h$). We can now calculate the number of oriented degree sequences with the given core fixed by each group element. The identity fixes every oriented degree sequence. Since there are $p$ ways to determine the first node and $p$ ways to determine the second node, the set has a total of $p^2$ elements. We see that $\chi(1) = p^2$. In order for an element to be fixed by $r_{180}$ it must be true that $A = D$ and $B = C$. There are $p$ ways to choose $A$ which determines the value of $D$. The value of $A$ also determines the value of $B$ (since $A + B = p - 1$) which determines the value of $C$. Then each value selected for $A$ creates exactly one oriented degree sequence that is fixed by $r_{180}$, so $\chi(r_{180}) = p$ for all values of $p$. Similarly we can consider $f_v$ which requires that $A = C$ and $B = D$. Parallel to the previous argument, we conclude that $\chi(f_v) = p$ for all values of $p$. Finally, we consider the trees that remain invariant under $f_h$. In this case, it must be true that $A = B$ and $C = D$. It is easy to see that $A = B$ occurs exactly once when $p - 1$ is even and zero otherwise. Thus, $\chi(f_h) = 1$ if $p - 1$ is even, and $\chi(f_h) = 0$ otherwise. We summarize these results below.
\(\chi(1) = p^2\) for all \(p\)

\(\chi(r_{180}) = p\) for all \(p\)

\(\chi(f_v) = p\) for all \(p\)

\(\chi(f_h) = 1\) if \((p - 1)\) is even

\(\chi(f_h) = 0\) if \((p - 1)\) is odd

Using Burnside’s theorem, we develop two cases by combining the above results. The following formulas count the number of orbits of the second structure given a particular value of \(p\).

\[
\zeta_2 = \frac{1}{4}(p^2 + 2p + 1) \quad \text{if } p \text{ is odd}
\]

\[
\zeta_2 = \frac{1}{4}(p^2 + 2p) \quad \text{if } p \text{ is even}
\]

Since the two tree structures examined completely partition the orbit possibilities for the column of \(k = 4\) we can add the results of each structure to determine the total number of orbits, \(\zeta\), for a given value of \(p\). Due to the conditions on the first tree structure, we must use six conditions to describe our results. The addition of the two cases yields the following result.
\[ \zeta = \frac{1}{3}[p^2 + 2p] \quad \text{if } p \equiv 0 \pmod{6} \]
\[ \zeta = \frac{1}{3}[p^2 + 2p] \quad \text{if } p \equiv 1 \pmod{6} \]
\[ \zeta = \frac{1}{3}[p^2 + 2p + 1] \quad \text{if } p \equiv 2 \pmod{6} \]
\[ \zeta = \frac{1}{3}[p^2 + 2p] \quad \text{if } p \equiv 3 \pmod{6} \]
\[ \zeta = \frac{1}{3}[p^2 + 2p] \quad \text{if } p \equiv 4 \pmod{6} \]
\[ \zeta = \frac{1}{3}[p^2 + 2p + 1] \quad \text{if } p \equiv 5 \pmod{6} \]

This result provides a formula which yields the values in the \( k = 4 \) column of the table for a given \( p \) value, thus proving the theorem. Notice that these results agree with the output of the program shown in Table 1.

4.5 The Column \( k = 5 \)

The column for \( k = 5 \) will be examined in order to develop an expression that can be used to calculate the number of orbits for a given \( p \) value. Recall that in regards to the generalized Catalan numbers, the notation \((p, k)\) indicates that the polygon to be divided will have \((p - 1)k + 2\) sides. This polygon will be divided into \( k \) polygons with \( p + 1 \) sides. In the case of \( k = 5 \), the polygon will be divided into five smaller polygons; this will be accomplished by drawing \((k - 1) = 4\) diagonals. Consistency in the number of sub-polygons formed allows for the problem to be easily partitioned.
using tree structures. It can be seen that a tree of five nodes (corresponding to the five sub-polygons) can have three distinct configurations, shown below.

![Tree Structures with Five Nodes](image)

**Figure 4.13: Tree Structures with Five Nodes**

It is useful to note that each tree structure in the column for \( k = 5 \) corresponds to the set of polygons with a certain number of key sections. In future cases, the number of key sections of a polygon may not be unique to exactly one tree structure. The partition with respect to tree structures will be used to calculate the total number of orbits for a given \( p \) value when \( k = 5 \). Considering each tree structure (in this case equivalent to each possible number of key sections), we will count the number of orbits in such a case. Using the partition, these numbers can be summed to calculate the total number of orbits.
Theorem 4.5. For \( k = 5 \) the number of distinct orbits, \( \zeta \) for a given \( p \) value is

\[
\zeta = \frac{1}{48} \left[ 25p^3 - 4p \right] \quad \text{if } p \text{ is even}
\]

\[
\zeta = \frac{1}{48} \left[ 25p^3 + 29p - 6 \right] \quad \text{if } p \equiv 1 \pmod{4}
\]

\[
\zeta = \frac{1}{48} \left[ 25p^3 + 29p + 6 \right] \quad \text{if } p \equiv 3 \pmod{4}.
\]

Proof. Although there are three distinct cases above, concerning the residue of \( p \) modulo 4, these are not the cases which will be immediately considered in the proof. We will instead first consider the cases of tree structure and secondly consider the sub-cases which consider the parity of \( p \). First, the tree structure that resembles a cross will be considered. In this tree structure, four of the five nodes are fully occupied by exterior edges of the polygon (represented by squares). Only the middle node can accept additional edges of the tree. Because of the size of the subpolygons, each node must have \((p + 1)\) edges extending from it. Since the center node already has four connections, \((p + 1) - 4 = (p - 3)\) connections must be added to the center node. There are four places in which these connections may be added, labeled \(a, b, c,\) and \(d\) in Figure 4.14.

We can see that these places form a square at the center of the tree structure. Because of the relation between tree structures and polygons, the symmetries of the
square correspond to the symmetries in the original polygon. In order to count the number of unique orbits, we need to account for and eliminate these symmetries and rotations that will create duplicates in the count. Using the structure of the square, we can see that any element of the dihedral group of order eight ($D_8$) could possibly result in an equivalent orbit. Thus, we will utilize Burnside’s Lemma. We now examine each element of $D_8$ to determine $\chi(g)$.

Note that $\chi(1)$ counts the total number of elements in the set of possible tree structures. These can be represented by compositions of length four, $C_{[4][p-3]}$, which can be computed as the number of permutations of a multiset, yielding $\frac{1}{6}(p^3 - 3p^2 + 2p)$ by lemma 4.3. For an element to be counted in $\chi(r_{90})$ and $\chi(r_{270})$ it is required that $a = b = c = d$, which only occurs when the sum of $a, b, c$ and $d$ is divisible by 4. When the sum is divisible by four there is exactly one way for $a = b = c = d$. Thus $\chi(r_{90}) = \chi(r_{270}) = 1$ if $4 | (p - 3)$ and $\chi(r_{90}) = \chi(r_{270}) = 0$ otherwise.
An element is fixed by $r_{180}$ if and only if $a = d$ and $b = c$. In order for this to be achieved, it must be true that $(p - 3)$ is divisible by two. Since each side of the structure (for example $a$ and $c$) must sum to $\frac{p-3}{2}$, we are able to develop a formula.

We determine the number of ways to select the first side by using a composition of length two, and note that $\chi(r_{180}) = C_{[2][\frac{p-3}{2}]} = \frac{p-1}{2}$. This composition determines the values of $a$ and $c$ which then determine the remaining values of the tree structure (since $a = d$ and $b = c$). Thus, the compositions describe all set elements that are fixed by the rotation of $180^\circ$. Similar to $\chi(r_{180})$, we see that $f_h$ fixes a polygonization only when $a = c$ and $b = d$, requiring that $(p-3)$ is divisible by two. We can approach the calculation exactly as above to yield the result $\chi(f_h) = \chi(f_v) = C_{[2][\frac{p-3}{2}]} = \frac{p-1}{2}$.

Finally, the most complex case is that of the diagonal flip elements. For an element to be fixed by $f_l$ or $f_r$ there must be a diagonal on which the two entries are equal. That is, either $a = d$ or $b = c$ must be true. Since the flips act identically, we will only consider the case where the line of symmetry goes through $c$ and $b$ and it is desired that $a = d$. In order for $a = d$, it must be true that $p - 3 - (c + b)$ is even.

We now consider two cases based on the parity of $p$.

First we assume that $p$ is odd. If $p$ is odd then $p-3$ is even. Thus, in order for $p - 3 - (c + b)$ to be even, it must be true that $(c + b)$ is even. Since we know that $(c + b)$ is selected from zero through $p - 3$, there are $\frac{p-3}{2}$ potential even values for $(c + b)$. We now must look at each even number to determine how many values of $(c, b)$ can be created such that $c$ and $b$ sum to the desired even number. We begin with zero. In order for $c + b = 0$ it is true that $c = 0$ and $b = 0$ which yields exactly one arrangement.
Next we consider $c + b = 2$. In this case there are three arrangements, $(2, 0), (0, 2)$ and $(1, 1)$. We are considering the number of compositions of $2n$ of length two where $0 \leq n \leq \frac{p-3}{2}$. Thus, we see that $\chi(f_l) = 1 + 3 + 5 + \ldots + (p - 2) = \frac{1}{4}(p^2 - 2p + 1)$, and similar for $\chi(f_r)$ when $p$ is odd.

If $p$ is even then $p - 3$ is odd. Thus, in this case we wish to consider the possible arrangements such that $c + b$ is an odd number. There are $\frac{p-2}{2}$ odd numbers which can be subtracted from $p-3$ to create an even number. We examine the number of arrangements for a given odd number starting with one. In order for $c + b = 1$ it can be true that $c = 1$ and $b = 0$ or $c = 0$ and $b = 1$, giving two arrangements. If $c + b = 3$ then there are four possible arrangements - $(3, 0), (0, 3), (2, 1)$ and $(1, 2)$. We are considering the number of compositions of $2n$ of length two where $0 \leq n \leq \frac{p-3}{2}$. Thus, in the case when $p$ is even we are considering the summation $2 + 4 + 6 + \ldots$, and there are $\frac{p-2}{2}$ summands. The result of this summation is $\frac{1}{4}(p^2 - 2p)$, and thus $\chi(f_l) = \chi(f_r) = \frac{1}{4}(p^2 - 2p)$ if $p$ is even.

Having considered each element of the dihedral group $D_8$, we summarize our results for this tree structure below.
\(\chi(1) = \frac{1}{6}(p^3 - 3p^2 + 2p)\) for all \(p\)

\(\chi(r_{90}) = \chi(r_{270}) = 1\) if \(4 \mid (p - 3)\)

\(\chi(r_{90}) = \chi(r_{270}) = 0\) if \(4 \nmid (p - 3)\)

\(\chi(r_{180}) = \frac{p - 1}{2}\) if \(2 \mid (p - 3)\)

\(\chi(r_{180}) = 0\) if \(2 \nmid (p - 3)\)

\(\chi(f_h) = \chi(f_v) = \frac{p - 1}{2}\) if \(2 \mid (p - 3)\)

\(\chi(f_h) = \chi(f_v) = 0\) if \(2 \nmid (p - 3)\)

\(\chi(f_r) = \chi(f_i) = \frac{1}{4}(p^2 - 2p + 1)\) if \(p\) is odd

\(\chi(f_r) = \chi(f_i) = \frac{1}{4}(p^2 - 2p)\) if \(p\) is even

It can be seen that the tree structure shaped like a cross introduces the three residues of \(p\) modulo 4 in the statement of the theorem. Values of \(p\) are partitioned into even \(p\) values, values of \(p\) in which \((p - 3)\) is divisible by two, and values of \(p\) where \((p - 3)\) is divisible by four. By selecting a subset of the values of \(p\), the number of orbits of the first tree structure can be determined by summing the appropriate \(\chi\) values and dividing by eight as Burnside’s theorem dictates. This summation yields the following result for the first tree structure, where \(\zeta_1\) represents the number of distinct orbits.
\[ \zeta_1 = \frac{1}{48} \left[ p^3 - 4p \right] \quad \text{if } p \equiv 0, 2 \pmod{4} \]

\[ \zeta_1 = \frac{1}{48} \left[ p^3 + 5p - 6 \right] \quad \text{if } p \equiv 1 \pmod{4} \]

\[ \zeta_1 = \frac{1}{48} \left[ p^3 + 5p + 6 \right] \quad \text{if } p \equiv 3 \pmod{4} \]

Next we will consider the case of the core tree structure with three key nodes and two interior nodes, as depicted and labeled in Figure 4.15. The three key nodes of the tree, denoted by squares, already have the appropriate number of connecting edges. Only the two center nodes are able to accept more connections. The diagram below depicts the labels which will be used in this section of the proof.

![Tree Structure with Three Key Nodes](image)

Figure 4.15: Tree Structure with Three Key Nodes

In the previous tree structure we utilized Burnside’s lemma with a dihedral group. In the examination of this tree structure we will again use Burnside’s lemma, this time creating a new group that deals with the tree structure in question. The group which will be created will consist of exactly the symmetries of the tree structure. Thus, the group contains only the identity element and the flip through the vertical
axis, \( f_v \). We will determine the number of set elements fixed by each group element before combining our results under Burnside’s lemma.

Since the number of connections of each node must equal \((p + 1)\), it can be seen that \( a + b + c = (p - 2) \) and \( d_1 + d_2 = (p - 1) \). In order to determine the number of set elements fixed by the identity, we must count all possible arrangements of the tree structure. Since \( d_1 + d_2 = (p - 1) \) we can determine that there are \( p \) ways for this node, referred to as the tail node, to be chosen. We arrive at this conclusion by noting that the choice of \( d_1 \) completely determines the value of \( d_2 \). Since \( d_1 \) is selected from the values zero through \( p - 1 \), there are exactly \( p \) possibilities for the tail node of the tree structure. The number of arrangements of the root node, defined as the node with three existing connections, is simply the compositions of \( p - 2 \) of length three, \( C_{[3]}[p-2] \), and thus, we conclude that \( \chi(1) = \frac{1}{2}p(p^2 - p) = \frac{1}{2}(p^3 - 2p) \) for all values of \( p \).

Next we determine the number of elements fixed by the flip through the vertical axis. We first observe that for any element fixed by the flip it is true that \( d_1 = d_2 \) and thus it must be true that \( p - 1 \) is even and \( p \) is odd. For an odd value of \( p \), the condition \( d_1 = d_2 \) occurs exactly once. In consideration of the root node, we note that we must determine the number of values of \( c \) such that \( a = b \). Since \( a \) must equal \( b \), then \( a + b \) is even. We see that alternating values of \( c \) will yield an even sum of \( a \) and \( b \). Since there are \( p - 1 \) options for \( c \), we can express this value as \( \frac{p - 1}{2} \) (an integer since \( p - 1 \) is even). Therefore, \( \chi(f_v) = \frac{1}{2}(p - 1) \) for odd values of \( p \) and \( \chi(f_v) = 0 \) for even values of \( p \).
We now summarize our results and combine them using Burnside’s lemma with a group of order two.

\[
\chi(1) = \frac{1}{2}(p^3 - p^2) \quad \text{for all values of } p
\]

\[
\chi(f_v) = \frac{1}{2}(p - 1) \quad \text{if } p \text{ is odd}
\]

\[
\chi(f_v) = 0 \quad \text{if } p \text{ is even}
\]

Letting \( \zeta_2 \) denote the number of orbits with this tree structure, we arrive at the following formula.

\[
\zeta_2 = \frac{1}{4}(p^3 - p^2 + p - 1) \quad \text{if } p \text{ is odd}
\]

\[
\zeta_2 = \frac{1}{4}(p^3 - p^2) \quad \text{if } p \text{ is even}
\]

We note that in the case of the tree structure with three key nodes only two residues are needed to consider all values of \( p \). Since the values of \( p \) are partitioned by parity in the case of three key nodes, it will be simple to combine the result with the previous tree structure which partitioned the values of \( p \) into even values and two types of odd values. The final tree structure that must be considered is the tree structure with only two key nodes. The tree that will be examined is labeled below.
To examine the final tree structure, we again utilize Burnside’s lemma. An alternative technique for counting the orbits of this tree structure can be found in Appendix A.

Using the knowledge that a node must have \((p+1)\) connections, it can be seen that \(a + b = c + d = e + f = p - 1\). We first note that three types of symmetries exist in this tree structure - the flip through the horizontal axis \((f_h)\), the flip through the vertical axis \((f_v)\) and the rotation of 180 degrees \((r_{180})\). It is easy to verify that these symmetries correspond to symmetries of the original polygon. In order to properly count the number of unique orbits, we must determine the number of tree structures fixed by each symmetry and the identity element.

We begin with the identity element of the symmetry group. The identity element fixes all possible tree structures. Since \(a + b = p - 1\) then there are \(p\) choices for the first node. The two remaining nodes also have \(p\) possibilities. Thus there are \(p^3\) different tree structures which can be formed, and \(\chi(1) = p^3\) for all values of \(p\).

In order for a tree to be fixed by \(f_h\) it must be true that \(a = b\), and \(c = d\) and \(e = f\). In order for this to be achieved it must be true that \(p - 1\) is even and thus \(p\) is odd. If \(p\) is odd then there is exactly one possibility for each node.
(a = b = c = d = e = f = \frac{p-1}{2})$. Therefore, $\chi(f_h) = 1$ if $p$ is odd, and $\chi(f_h) = 0$ if $p$ is even.

In the case of the flip through the vertical axis the values of $c$ and $d$ are irrelevant since they lie on the axis of symmetry. Thus there are $p$ possibilities for this node. Additionally it must be true that $a = e$ and $b = f$. We begin by selecting the value of $a$ (of which there are $p$ possibilities). The value of $a$ determines the value of $e$ (since $a = e$) and the value of $b$ (since $a + b = p - 1$). The value of $b$ then determines the value of $f$ (since $b = f$). Thus there are $p$ ways to determine the remaining two nodes, and $\chi(f_v) = p^2$ for all values of $p$.

Finally we consider the rotation of 180 degrees. When the tree structure is rotated we see that the following equalities must hold: $a = f$, $b = e$ and $c = d$. Since $c + d = p - 1$ and $c = d$, it must be true that $p - 1$ is even and thus $p$ is odd. If $p$ is odd then there is exactly one possibility for the center node. Additionally, we select the value of $a$ from the initial $p$ possibilities. This determines the value of $b$ (since $a + b = p - 1$) as well as the value of $f$ (since $a = f$). The value of $f$ then determines the value of $e$ since $e + f = p - 1$. We are then able to conclude that $\chi(r_{180}) = p$ if $p$ is odd and $\chi(r_{180}) = 0$ if $p$ is even. We now summarize the results for the four group elements.
\[ \chi(1) = p^3 \quad \text{for all } p \]
\[ \chi(f_h) = 1 \quad \text{if } p \text{ is odd} \]
\[ \chi(f_h) = 0 \quad \text{if } p \text{ is even} \]
\[ \chi(f_o) = p^2 \quad \text{for all } p \]
\[ \chi(r_{180}) = p \quad \text{if } p \text{ is odd} \]
\[ \chi(r_{180}) = 0 \quad \text{if } p \text{ is even} \]

We combine the above results using Burnside’s lemma on a group of order four to obtain the following formula, where \( \zeta_3 \) is the number of orbits with this tree structure.

\[ \zeta_3 = \frac{1}{4}(p^3 + p^2 + p + 1) \quad \text{if } p \text{ is odd} \]
\[ \zeta_3 = \frac{1}{4}(p^3 + p^2) \quad \text{if } p \text{ is even} \]

Having examined each of the tree structures, we are now ready to combine and simplify our results for the \( k = 5 \) column. By summing the formulas developed in the previous discussion for a particular division of \( p \) values, we get the following result.
\[
\zeta = \frac{1}{48} \left[ 25p^3 - 4p \right] \quad \text{if } p \text{ is even}
\]
\[
\zeta = \frac{1}{48} \left[ 25p^3 + 29p - 6 \right] \quad \text{if } p \equiv (\text{mod } 4)
\]
\[
\zeta = \frac{1}{48} \left[ 25p^3 + 29p + 6 \right] \quad \text{if } p \equiv (\text{mod } 4)
\]

This result calculates the number of distinct orbits for \( k = 5 \) and a given value of \( p \), thus proving the theorem.

\[\square\]

4.6 The Column \( k = 6 \)

In the case of \( k = 6 \) we will approach our goal in a similar manner as in the \( k = 5 \) column. We will examine tree structures as an alternative to regular polygons. We begin by determining all possible core structures for the column \( k = 6 \). The generalized Catalan numbers dictate that a given polygon would be divided into \( k \) smaller sub-polygons. Using the bijection between regular polygons and tree structures, we determine that each core structure must have exactly \( k = 6 \) nodes present. Below we illustrate the core tree structures which must be considered in the proof of the subsequent theorem.
Figure 4.17: Core Tree Structures for the Column $k = 6$
Theorem 4.6. For \( k = 6 \) the number of distinct orbits, \( \zeta \), for a given \( p \) value is

\[
\zeta = \frac{1}{10} \left[ 9p^4 - 9p^3 + 14p^2 - 4p \right] \quad \text{if} \ p \equiv 0, 1, 2, 3, 5, 6, 7, 8 \ (\text{mod} \ 10)
\]

\[
\zeta = \frac{1}{10} \left[ 9p^4 - 9p^3 + 14p^2 - 4p + 4 \right] \quad \text{if} \ p \equiv 4 \ (\text{mod} \ 10)
\]

\[
\zeta = \frac{1}{20} \left[ 18p^4 - 18p^3 + 28p^2 - 8p + 8 \right] \quad \text{if} \ p \equiv 9 \ (\text{mod} \ 10).
\]

Proof. In order to develop the above formulae and thus provide a proof for the theorem, we will consider each of the core tree structures in the order in which they are numbered in Figure 4.17. In each case we will utilize Burnside’s Lemma on the symmetry group of the particular tree structure. This technique was used earlier in the proofs of the \( k = 4 \) and \( k = 5 \) columns. We begin by considering the first tree structure referred to as the straight line core which is depicted in detail below.

![Symmetries of the Straight Line Core](image_url)

Figure 4.18: Symmetries of the Straight Line Core

By examining the above illustration we can determine our symmetry group to be comprised of the identity element, a rotation of \( 180^\circ \), a flip through the vertical
axis \((f_v)\), and a flip through the horizontal axis \((f_h)\). To utilize Burnside’s Lemma we must determine the number of set elements fixed by each member of this symmetry group of size four.

Each interior node in the core (depicted by circles) must have a total of \((p + 1)\) connections. Since each interior node already has two existing connections, \((p - 1)\) additional connections are needed for each interior node. Thus, in terms of the illustration labels, it is true that \(a + b = c + d = e + f = g + h = p - 1\). We are then able to conclude that there are \(p\) ways to choose each node. Thus, there are exactly \(p^4\) ways to select the values \(a, b, \ldots, h\), and thus there are exactly \(p^4\) members of the set of polygonizations with this core. Since the group identity element fixes every member of the set, we conclude that \(\chi(1) = p^4\) for all values of \(p\).

Next we consider the horizontal flip. In order for an element to be fixed by the horizontal flip it must be true that \(a = b, c = d, e = f,\) and \(g = h\). Since \(a + b = p - 1\) then it must be true that \(p - 1\) is even and thus \(p\) is odd. In the case that \(p\) is odd there is exactly one pair of values such that \(a = b\). Thus there is one possibility for each node, yielding one fixed element when \(p\) is odd and zero fixed elements when \(p\) is even. Therefore, \(\chi(f_h) = 1\) if \(p\) is odd and \(\chi(f_h) = 0\) otherwise.

In the case of the vertical flip, we see that the choice of one side (that is, \(a, b, c, d\)) completely determines the remaining side. As stated above, there are exactly \(p\) possibilities for a given node. Thus there are \(p^2\) ways to select the left hand side of the tree structure. Since this completely determines the tree structure, we see that \(\chi(f_v) = p^2\) for all values of \(p\).
The final group element to be considered is the $180^\circ$ rotation ($r_{180}$). We will utilize the diagram below to count the number of fixed set elements under this rotation.

![Diagram showing the consequences of the $180^\circ$ rotation]

By examining the illustration, we see that if we select the value of $a$ (of which there are $p$ possibilities) then we completely determine the value of $b$ (since $a + b = p - 1$) as well as the values of $g$ and $h$ (since the rotation requires $a = h$ and $b = g$). Thus there are $p$ ways to select $a, b, g$ and $h$. Similarly, there are exactly $p$ ways to choose $c, d, e$ and $f$. Because of the independent nature of the nodes, we conclude that there are exactly $p^2$ elements fixed by $r_{180}$. Therefore, $\chi(r_{180}) = p^2$ for all values of $p$. Having examined all four group elements, we summarize our results below.
\[
\chi(1) = p^4 \quad \text{for all } p \\
\chi(f_h) = 1 \quad \text{if } p \text{ is odd} \\
\chi(f_h) = 0 \quad \text{if } p \text{ is even} \\
\chi(f_v) = p^2 \quad \text{for all } p \\
\chi(r_{180}) = p^2 \quad \text{for all } p
\]

We then utilize Burnside’s Lemma by summing the number of fixed set elements and then dividing by the order of the group to yield the following result, where \( \zeta_1 \) is the number of generalized Catalan orbits containing the straight line core.

\[
\zeta_1 = \frac{1}{4} \left( p^4 + 2p^2 \right) \quad \text{if } p \text{ is even} \\
\zeta_1 = \frac{1}{4} \left( p^4 + 2p^2 + 1 \right) \quad \text{if } p \text{ is odd}
\]

Next we consider the second core tree structure which can be referred to as the upside-down T-core. Figure 4.20 depicts the labeled upside-down T-core.

In the case of the upside-down T-core, we see that the symmetry group contains exactly two elements, the identity element and the flip through the vertical axis. We will first determine the total number of set elements and thus \( \chi(1) \). We note that \( a + b = c + d = p - 1 \) yielding \( p \) possible choices for each of these interior nodes.
For the remaining center node, it must be true that $e + f + g = p - 2$. This can be expressed as a composition of $p - 2$ of length three, $C_{[3][p-2]}$. Similar to Lemma 4.3, $C_{[3][p-2]}$ can be calculated using multisets, yielding $\frac{1}{2}(p^2 - p)$. To determine the total number of set elements we multiply the results from the three independent interior nodes, resulting in $\chi(1) = p^2 \cdot \frac{1}{2}(p^2 - p) = \frac{1}{2}(p^4 - p^3)$ for all values of $p$.

We next consider the elements fixed by $f_v$. It is clear that $a$ must equal $c$ and $b$ must equal $d$. By selecting a value for $a$ (of which there are $p$ possibilities) we can determine the values of $b$ (since $a + b = p - 1$), $c$ (since $a = c$) and $d$ (since $c + d = p - 1$ or since $b = d$). Thus, there are exactly $p$ ways to select the values of the two interior nodes in question. In the case of the center interior node, we note that $e + f + g = p - 2$. Since it must be the case that $e = f$, $p - 2 - g$ must be an even number. There are exactly $p - 1$ choices for the value of $g$ (0 through $p - 2$). The number of values which ensure that $p - 2 - g$ is even is $\lceil \frac{p-1}{2} \rceil$. We can remove the ceiling function by considering the parity of $p$. We obtain the following result: $\chi(f_v) = \frac{1}{2}p^2$ for even values of $p$ and $\chi(f_v) = \frac{1}{2}(p^2 - p)$ for odd values of $p$. 

Figure 4.20: Symmetries of the Upside-Down T-core
Having examined both group elements, we summarize our results below followed by
the utilization of Burnside’s Lemma on the symmetry group of order two. We define
\( \zeta_2 \) to be the number of generalized Catalan orbits containing the upside-down T-core.

\[
\chi(1) = \frac{1}{2}(p^4 - p^3) \\
\frac{1}{2}p^2 \quad \text{if } p \text{ is even} \\
\frac{1}{2}(p^2 - p) \quad \text{if } p \text{ is odd}
\]

\[
\zeta_2 = \frac{1}{4}(p^4 - p^3 + p^2) \quad \text{if } p \text{ is even} \\
\zeta_2 = \frac{1}{4}(p^4 - p^3 + p^2 - p) \quad \text{if } p \text{ is odd}
\]

We now examine the third tree structure, the cross core. In Figure 4.21 we
label the nodes of the core as well as the symmetries of the acting group.

The group which acts on the cross core has exactly two elements, the identity
element and the flip through the vertical axis. We begin by examining the elements
fixed by the identity element. As in previous cases, since \( e + f = p - 1 \), there are
exactly \( p \) ways in which the values of this node can be chosen. Next, we note that
\( a + b + c + d = p - 3 \). We utilize Lemma 4.3 directly to determine the value of this
composition of length four. By combining the results of the two nodes we determine
that \( \chi(1) = \frac{1}{6}(p^4 - 3p^3 + 2p^2) \) for all values of \( p \).
We next consider the flip through the vertical axis. In order for a set element to be fixed by \( f_v \) it must be true that \( e = f \). Thus, \( p - 1 \) must be even and \( p \) must be odd. In addition, being fixed by the flip ensures that \( a = b \) and \( c = d \) for \( a + b + c + d = p - 3 \). Therefore, it is needed that \( 2 | (p - 3) \). The selection of \( a \) determines the value of \( b \) (since \( a = b \)). It is then true that \( c + d = p - 3 - a - b \), where \( p - 3 - a - b \) is even since \( p - 3 \) is even and \( a + b \) is even. Thus, since it must be true that \( c = d \), \( c = d = \frac{p - 3 - a - b}{2} \). Since \( a \) is selected from the values zero through \( \frac{p - 3}{2} \), there are exactly \( \frac{p - 1}{2} \) possibilities. Then, \( \chi(f_v) = \frac{p - 1}{2} \) for odd values of \( p \) and \( \chi(f_v) = 0 \) otherwise.

We summarize the results below and combine them using Burnside’s Lemma with an acting group of order two. We define \( \zeta_3 \) to be the number of generalized Catalan orbits containing the cross core.
\[ \chi(1) = \frac{1}{6} \left( p^4 - 3p^3 + 2p^2 \right) \]  
for all \( p \)

\[ \chi(f_v) = \begin{cases} 
\frac{p-1}{2} & \text{if } p \text{ is odd} \\
0 & \text{if } p \text{ is even} 
\end{cases} \]

\[ \zeta_3 = \frac{1}{12} \left( p^4 - 3p^3 + 2p^2 \right) \]  
if \( p \) is even

\[ \zeta_3 = \frac{1}{12} \left( p^4 - 3p^3 + 2p^2 + 3p - 3 \right) \]  
if \( p \) is odd

The fourth tree structure is referred to as the Y-core, and is illustrated in Figure 4.22.

![Figure 4.22: Symmetries of the Y-core](image)

We will find that computations involving the Y-core will be very similar to arguments made in previous core structures. We begin by observing that the acting group has two elements - the identity element and the flip across the horizontal axis. To determine the number of set elements fixed by the identity we use previous
techniques to determine that each of the tail nodes (each with two connections) have exactly \( p \) arrangements. The base node (with three connections) is expressed by the composition of \( p - 2 \) of length three (as in the case of the upside down T-core) yielding \( \frac{1}{2}(p^2 - p) \). The multiplication of the independent nodes provides the result \( \chi(1) = \frac{1}{2}p^2(p^2 - p) \).

In examination of the horizontal flip, we note that \( d \) must equal \( e \), implying that \( p - 1 \) is even and thus \( p \) is odd. As before, for an odd value of \( p \) there is exactly one way for which \( d = e \) and one way for \( f = g \). Next the base node is examined. We arbitrarily select \( c \) from the values zero to \( p - 2 \) (exactly \( p - 1 \) possibilities). Since \( p \) is odd, exactly \( \frac{p-1}{2} \) of these selections yield an even result for \( a + b = p - 2 - c \) so that it is possible for \( a \) to equal \( b \). Thus, \( \chi(f_h) = \frac{p-1}{2} \) for odd values of \( p \) and \( \chi(f_h) = 0 \) for even values of \( p \).

These results are summarized below, followed by using Burnside’s Lemma with the symmetry group of order two, where \( \zeta_4 \) is defined to be the number of generalized Catalan orbits containing the Y-core.

\[
\chi(1) = \frac{1}{2} (p^4 - p^3)
\]

\[
\chi(f_h) = \begin{cases} 
\frac{p-1}{2} & \text{if } p \text{ is odd} \\
0 & \text{if } p \text{ is even}
\end{cases}
\]
\[ \zeta_4 = \frac{1}{4} [p^4 - p^3] \quad \text{if } p \text{ is even} \]
\[ \zeta_4 = \frac{1}{4} [p^4 - p^3 + p - 1] \quad \text{if } p \text{ is odd} \]

The double Y-core is the fifth tree structure, which we illustrate in Figure 4.23.

Figure 4.23: Symmetries of the Double Y-core

The group which acts on the double Y-core has four elements. The two flips, \( f_v \) and \( f_h \), are depicted in the above illustration. Additional elements are the identity element and the rotation of \( 180^\circ \), denoted \( r_{180} \). Starting with the identity element we observe that \( a + b + c = p - 2 \) and similarly, \( d + e + f = p - 2 \). To count all of the possible arrangements of each node we calculate the number of compositions of \( p - 2 \) of length 3, \( C_{[3]}[p-2] \). Calculating the composition with the technique of multisets yields \( \frac{1}{2}(p^2 - p) \). As both nodes have the same structure, the number of arrangements for each is expressed by this formula. By multiplying the independent results we obtain \( \chi(1) = \frac{1}{4}(p^4 - 2p^3 + p^2) \) for all values of \( p \).
Next we examine the flip through the vertical axis, $f_v$. We note that one side of the double Y-core must be congruent to the other side of the core in order for a set element to be fixed under this symmetry. Thus the selection of one side of the core must completely determine the values of the other side. There are exactly $\frac{1}{2}(p^2 - p)$ ways to select the first side and thus $\chi(f_v) = \frac{1}{2}(p^2 - p)$ for all values of $p$.

Since the rotation of $180^\circ$ effectively switches the placement of either side of the double Y-core, it acts in the same manner as the vertical flip. By selecting one side of the core we completely determine the other side. Therefore, $\chi(r_{180}) = \frac{1}{2}(p^2 - p)$ for all values of $p$.

In the case of $f_h$ we note that both nodes will act in identical manners in relation to the flip through the horizontal axis. Thus we examine one node and note that our result will also apply to the remaining independent node. We begin by selecting a value for $c$ since it is unaffected by the symmetry. Having selected $c$, it must be true that $a + b = p - 2 - c$ is even in order to achieve $a = b$. Since $c$ is chosen from the values zero through $p - 2$ (of which there are $p - 1$ unique values) then exactly $\lceil \frac{p-1}{2} \rceil$ of these values yield an even result for $a + b$. By considering the parity of $p$ we are able to remove the ceiling function resulting with $\chi(f_h) = (\frac{p}{2})^2 = \frac{1}{4}p^2$ for even values of $p$ and $\chi(f_h) = (\frac{p-1}{2})^2 = \frac{1}{4}(p^2 - 2p + 1)$ for odd values of $p$.

The results of the four group elements are summarized below followed by the results yielded by Burnside’s Lemma. We define $\zeta_5$ to be the number of generalized Catalan orbits containing the double Y-core.
\[ \chi(1) = \frac{1}{4} (p^4 - 2p^3 + p^2) \quad \text{for all } p \]

\[ \chi(f_v) = \frac{1}{2} (p^2 - p) \quad \text{for all } p \]

\[ \chi(r_{180}) = \frac{1}{2} (p^2 - p) \quad \text{for all } p \]

\[ \chi(f_h) = \begin{cases} 
\frac{1}{4} p^2 & \text{if } p \text{ is even} \\
\frac{1}{4} (p^2 - 2p + 1) & \text{if } p \text{ is odd} 
\end{cases} \]

\[ \zeta_5 = \frac{1}{16} \left[ p^4 - 2p^3 + 6p^2 - 4p \right] \quad \text{if } p \text{ is even} \]

\[ \zeta_5 = \frac{1}{16} \left[ p^4 - 2p^3 + 6p^2 - 6p + 1 \right] \quad \text{if } p \text{ is odd} \]

The final tree structure, the star core, is depicted in Figure 4.24 with one of the five symmetries labeled.

By examining the five key nodes of the core (represented by squares) we see that the general shape formed by the core is that of a pentagon. The corresponding symmetry group which acts upon this pentagon-shaped core is in fact the dihedral group of order ten (known as \( D_{10} \)). The group consists of the identity element, four rotations and five flips (one of which is shown above). Because of the nature of the core, it is reasonable to consider only one rotation and one flip as the others will act in an identical manner. Thus, we must consider the number of elements fixed by
the identity, the rotation \( (r) \), and the flip \( (f) \). The number of set elements fixed by
the identity is the number of compositions of \( p - 4 \) of length five \( (C_{[5]}[p-4]) \), since
\[ a + b + c + d + e = p - 4. \]
By again utilizing the technique of the counting of multisets we obtain
\[ \chi(1) = \frac{1}{24}(p^4 - 6p^3 + 11p^2 - 6p) \]
for all values of \( p \).

Next we consider the rotation. In order for an element to be fixed under one of the four rotations, every value \( (a, \ldots, e) \) must be equal. In order for all values to be equal it must be true that \( 5 | (p - 4) \). Therefore, \( \chi(r) = 1 \) if \( 5 | (p - 4) \) and \( \chi(r) = 0 \) otherwise.

In consideration of the flip, we begin by selecting the value which is not affected by the flip. Using the labels above, we select the value of \( e \) (note that there are \( p - 3 \) choices for \( e \)). Once \( e \) has been selected we must consider the number of ways that \( a + b + c + d \) can be chosen such that \( a = b \) and \( c = d \). It is clear that this can only occur when \( a + b + c + d \) is even. If \( p \) is even then there exist \( \frac{p-2}{2} \) values of \( e \) such that \( a + b + c + d \) is even and \( \frac{p-3}{2} \) values of \( e \) if \( p \) is odd. Now we must determine
how many arrangements are fixed under the flip for each value of \( e \). Figure 4.25 gives an example for the case of \( p = 12 \) and thus \( p - 4 = 8 \).

From this example we see that the number of elements which are fixed for a given value of \( e \) is given by the common sequence 1, 2, 3, ... which terminates when all possible values of \( e \) have been considered. Since we would like the total number of values fixed by the flip (across all values of \( e \)) we can sum these results using the known total number of \( e \) values calculated above. We compute the sum below.

\[
\chi(f) = \sum_{i=1}^{\frac{p-2}{2}} i = \frac{1}{8}(p^2 - 2p) \quad \text{for even values of } p
\]

\[
\chi(f) = \sum_{i=1}^{\frac{p-3}{2}} i = \frac{1}{8}(p^2 - 4p + 3) \quad \text{for odd values of } p
\]

We summarize our results for the group elements below. We are then able to combine these results taking into account that there are five flips which act identically.
and four rotations which also act identically. We define $\zeta_6$ to be the number of generalized Catalan orbits containing the star core.

$$\chi(1) = \frac{1}{24} \left( p^4 - 6p^3 + 11p^2 - 6p \right)$$

for all $p$

$$\chi(r) = \begin{cases} 
1 & \text{if } 5 | (p - 4) \\
0 & \text{otherwise} 
\end{cases}$$

$$\chi(f) = \begin{cases} 
\frac{1}{8} (p^2 - 2p) & \text{if } p \text{ is even} \\
\frac{1}{8} (p^2 - 4p + 3) & \text{if } p \text{ is odd} 
\end{cases}$$

$$\zeta_6 = \frac{1}{240} \left[ p^4 - 6p^3 + 26p^2 - 36p \right]$$

if $p \equiv 0, 2, 6, 8 \pmod{10}$

$$\zeta_6 = \frac{1}{240} \left[ p^4 - 6p^3 + 26p^2 - 66p + 45 \right]$$

if $p \equiv 1, 3, 5, 7 \pmod{10}$

$$\zeta_6 = \frac{1}{240} \left[ p^4 - 6p^3 + 26p^2 - 36p + 96 \right]$$

if $p \equiv 4 \pmod{10}$

$$\zeta_6 = \frac{1}{240} \left[ p^4 - 6p^3 + 26p^2 - 66p + 141 \right]$$

if $p \equiv 9 \pmod{10}$

Now that we have considered each core for the case of $k = 6$ we are able to build a general formula for the $k = 6$ column. Since we have developed formulae for each core, we are able to combine our previous results to develop the general formula. By allowing $\zeta$ to be defined as the sum of $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$ and $\zeta_6$ we obtain the following result.
\[ \zeta = \frac{1}{10} \left[ 9p^4 - 9p^3 + 14p^2 - 4p \right] \quad \text{if } p \equiv 0, 1, 2, 3, 5, 6, 7, 8 \pmod{10} \]

\[ \zeta = \frac{1}{10} \left[ 9p^4 - 9p^3 + 14p^2 - 4p + 4 \right] \quad \text{if } p \equiv 4 \pmod{10} \]

\[ \zeta = \frac{1}{20} \left[ 18p^4 - 18p^3 + 28p^2 - 8p + 8 \right] \quad \text{if } p \equiv 9 \pmod{10} \]

This proves the theorem.

4.7 Future Research

There are many aspects of the generalized Catalan orbits which could potentially be studied in the future. The columns could be continued to be studied in hopes of revealing a general trend in the resulting formulas. Additionally, research could expand on the study of the rows which was done in [5]. It is likely that an asymptotic bound could be discovered for the number of generalized Catalan orbits. Furthermore, the nature of the orbits could be analyzed. The rank of each orbit could be computed, and the number of regular orbits could be counted.
BIBLIOGRAPHY


APPENDIX

AN ALTERNATIVE TECHNIQUE ILLUSTRATED WITH $K = 5$

It is often the case that the utilization of Burnside’s lemma is the most straightforward and simplest way to compute the number of generalized Catalan orbits; however, it is useful to know a variety of techniques that can be used to count the orbits. In this section we will consider an alternative way to count the number of generalized Catalan orbits given a particular tree structure. We will illustrate the method using the third tree structure from the case of $k = 5$. The tree structure to be considered is illustrated and labeled in Figure A.1. The technique which we will use will consider the total number of arrangements of the tree structure and subtract the number of arrangements that are duplicated because of symmetries of the tree.

![Figure A.1: Tree Structure with Two Exterior Nodes](image)

Using the knowledge that a node must have $(p+1)$ connections, it can be seen that $a + b = c + d = e + f = p - 1$. Thus there are a total of $p^3$ arrangements which include the duplications of orbits due to symmetries of the tree structure. We first
note that three types of symmetries exist in the tree structure - the flip through the horizontal axis, the flip through the vertical axis and the rotation of $180^\circ$. It is easy to verify that these symmetries correspond to symmetries of the original polygon. In order to properly count the number of unique orbits, these symmetries must be considered. We will begin the counting process by selecting a node that will allow us to reduce the number of duplications which we must consider. Next we will consider the duplications caused by the symmetries of the tree. For simplification, we begin by partitioning the values of $p$ by parity. This will help to isolate specific cases where duplicate orbits are generated.

We begin by considering even values of $p$. Although there are $p^3$ total possibilities (as noted previously) we will begin with a different total in order to immediately eliminate the consequences of one symmetry. We begin by considering a partition of the middle node (as opposed to the composition that is used to develop the $p^3$ arrangements), yielding $P_{[2][p-1]} = \frac{p}{2}$ by Lemma 4.1. The use of a partition will eliminate the duplications caused by the flip through the horizontal axis ($f_h$) since the tree structure has been fixed in position by the partition and it is never the case that $c = d$ (since $(p - 1)$ is odd). Additionally, this selection removes duplicates caused by the $180^\circ$ rotation since under $r_{180}$ the places of $c$ and $d$ are switched. Thus, our first node (the center node) has $\frac{1}{2}p$ possibilities.

Next we account for duplicates caused by vertical symmetries - this portion will utilize the subtraction of duplications. If we begin by counting the total number of generated orbits (ignoring the chosen values of $c$ and $d$ which were handled above)
using compositions, we see that the total ways to choose $a, b, e$ and $f$ is $(\binom{p}{2})^2 = p^2$. From this number we must subtract the duplications. We note that the case where $a = e$ and $b = f$ is generated exactly once since there is only one configuration that may be chosen from each composition. We must be careful to exclude this case when subtracting duplicates. We begin by selecting the value of $a$, of which there are $p$ possibilities. We then select the value of $e$ while excluding the single case where $a = e$, and this yields $p - 1$ possibilities. Finally, we divide by two since half of the resulting orbits are duplicated because of the vertical line of symmetry. Below we illustrate an example of arrangements which are in the same orbit because of the vertical symmetry.

![Figure A.2: Two Congruent Orbits in the Case of Two Exterior Nodes with $p = 4$](image)

Thus, we conclude that there are $\binom{p}{2} = p$ ways to choose the first pair of elements and $\frac{\binom{p-1}{2}}{2} = \frac{p-1}{2}$ ways to choose the second pair of elements such that a duplicate orbit is generated. We will subtract this from the total number of ways to select the two nodes in question. Then, the total number of unique orbits can be expressed and simplified in the following way.
\[ \zeta_3 = P_{[2][p-1]} \left[ C_{[2][p-1]}^2 - \left( \frac{C_{[2][p-1]} - 1}{2} \right) \left( C_{[2][p-1]} \right) \right] \]

\[ = \frac{1}{2}p \left[ p^2 - \frac{1}{2}p(p - 1) \right] \]

\[ = \frac{1}{4} \left[ p^3 + p^2 \right] \text{ if } p \text{ is even} \]

Now we examine the case where \( p \) is odd. If \( p \) is odd then \((p - 1)\) is even, and it becomes more difficult to eliminate the horizontal symmetry of the tree structure since it is now possible that \( c = d \). To handle this case, we utilize the formula above. By substituting \( (P_{[2][p-1]} - 1) \) in place of \( P_{[2][p-1]} \), we can effectively remove the case of \( c = d \) and calculate the number of orbits of this case separately. Then, our formula will have the following form.

\[ \zeta = (P_{[2][p-1]} - 1) \left[ C_{[2][p-1]}^2 - \left( \frac{C_{[2][p-1]} - 1}{2} \right) \left( C_{[2][p-1]} \right) \right] \]

\[ + \text{ the number of unique orbits in the case of } c = d \]

In order to determine the number of orbits for the case when \( c = d \), we examine a small example using \( p = 5 \).

By examining Figure A.3, it can be seen that the number of generalized Catalan orbits where \( c = d \) can be expressed by the summation \( p + (p - 2) + ... + 1 \). The number of summands is equivalent to the number of partitions of \((p - 1)\),
Figure A.3: Orbits for the Case of $c = \text{when } p = 5$

$P_{[2][p-1]} = \frac{p+1}{2}$. To form the summation as seen in the example, we select one of the possible partitions of $(p - 1)$ to fill the left node. We then use each of the $p$ possible compositions to fill the right node. We then select another distinct partition of $(p - 1)$ for the left node. In order to fill the right node we select from the compositions of $(p - 1)$ which do not include either ordering of the composition previously used for the left node (of which there are $(p - 2)$ possibilities). By removing these possibilities we eliminate configurations which fall into the same orbit. We continue this process until we have exhausted all possible compositions for the left node. Thus the total number of unique orbits in the case where $p$ is odd is given and simplified below.

$$\zeta = (P_{[t=2][p-1]} - 1) \left[ C_{[t=2][p-1]}^2 - \left( \frac{C_{[t=2][p-1]}}{2} \right) (C_{[t=2][p-1]}) \right]$$

$$+ [p + (p - 1) + \ldots + 1]$$

$$= \frac{1}{4} \left[ p^3 + p^2 + p + 1 \right]$$

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We can see that the above technique yields the same results which we developed in the proof of Theorem 4.5.