Instructions: Test #1 will be Monday, February 9, 2004. The test covers the following sections in the text: §§2.1-2.6; §§3.1,3.2. The following are a selection of problems from the material to be covered on the test. These problems do not represent the entirely of the types of problems that may appear on the test. Solve these problems, ideally, without reference to your text. On Saturday at 5:00 pm, I will upload the solutions to these problem for you to compare with your own solutions.

1. True or False. Insert a T (for True) or a F (for False) into the blanks below in response to each of the assertions.

(a) **F** If \( f \) is continuous at \( x = a \), then \( f \) must have a tangent line at \((a, f(a))\).

*Solution:* Not true (false), for example, \( f(x) = |x| \) is continuous at \( x = 0 \), but is not differentiable there. It the converse statement that is true: If \( f \) is differentiable at \( x = a \), then \( f \) is continuous at \( x = a \).

(b) **F** If \( a \) is not in the domain of the function \( f \), then the \( \lim_{x \to a} f(x) \) cannot exist.

*Solution:* False. For example, \( f(x) = \frac{x}{x} \), for \( x \neq 0 \). But \( \lim_{x \to 0} f(x) = 1 \).

(c) **T** The function \( f(x) = \begin{cases} x^2 - 1 & x < -2 \\ 1 - 2x & x \geq -2 \end{cases} \) has a jump discontinuity at \( x = -2 \)

*Solution:* It is easy to calculate \( \lim_{x \to -2^+} f(x) = 3 \) and \( \lim_{x \to -2^-} f(x) = 5 \). So \( \lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x) \); this means \( f \) has a jump discontinuity at \( x = -2 \).

2. Use the precise definition of limit to prove that \( \lim_{x \to -1} (4x - 3) = -7 \).

*Solution:* Let \( \epsilon > 0 \), we want to find a \( \delta > 0 \) such that \( 0 < \left| x + 1 \right| < \delta \implies \left| (4x - 3) - (-7) \right| < \epsilon \). To this end, let’s examine the inequality \( \left| (4x - 3) - (-7) \right| < \epsilon \). Now, manipulating this inequality we obtain:

\[
\left| (4x - 3) - (-7) \right| < \epsilon \iff \left| 4x + 4 \right| < \epsilon \iff 4 \left| x + 1 \right| < \epsilon \iff \left| x + 1 \right| < \frac{\epsilon}{4}
\]

This suggests that we should choose \( \delta = \epsilon/4 \).

Indeed, let \( \delta = \epsilon/4 \). Now,

\[
0 < \left| x + 1 \right| < \delta \implies \left| x + 1 \right| < \frac{\epsilon}{4} \]

\[
\implies 4 \left| x + 1 \right| < \epsilon \]

\[
\implies \left| 4x + 4 \right| < \epsilon \]

\[
\implies \left| (4x - 3) - (-7) \right| < \epsilon
\]

which is what we wanted to prove.

3. Compute the limit of each of the following.

(a) \( \lim_{x \to -1} \frac{2x^3 - 3x - 4}{2 - 3x} \)
Solution: This problem is of Skill Level 0:

\[ \lim_{x \to -1} \frac{2x^3 - 3x - 4}{2 - 3x} = \frac{2(-1)^3 - 3(-1) - 4}{2 - 3(-1)} = \frac{-2 + 3 - 4}{2 + 3} = \frac{-3}{5} \]

(b) \( \lim_{x \to 1} \frac{x - 1}{x^2 - 3x + 2} \)

Solution: As the limit of the denominator is zero, we must use other methods:

\[ \lim_{x \to 1} \frac{x - 1}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x - 2)} = \lim_{x \to 1} \frac{1}{x - 2} = -1 \]

(c) \( \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} \)

Solution:

\[ \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} = \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} \cdot \frac{3 + \sqrt{t}}{3 + \sqrt{t}} \quad \text{multiply by conjugate} \]

\[ = \lim_{t \to 9} \frac{(9 - t)(3 + \sqrt{t})}{3 - t} \quad \text{simplify} \]

\[ = \lim_{t \to 9} 3 + \sqrt{t} \quad \text{cancel} \]

\[ = 6 \quad \text{skill level 0} \]

(d) \( \lim_{x \to 3^-} \frac{4x^2 - 2}{3 - x} \)

Solution: As \( x \to 3^- \), we have \( x < 3 \) and getting closer and closer to 3. But \( x < 3 \) implies \( 3 - x > 0 \). Notice also, that when \( x \) is close to 3, the numerator is close to 34 which is positive. Therefore, for \( x \) close to 3, both numerator and denominator are positive. But the denominator goes to zero, while the numerator goes to 34, therefore,

\[ \lim_{x \to 3^-} \frac{4x^2 - 2}{3 - x} = +\infty \]
4. Let \( f(x) = \begin{cases} 3x - 2 & x \leq -2 \\ 4x^2 & x > -2 \end{cases} \)

(a) Compute \( \lim_{x \to -2^+} f(x) \)

Solution: We have
\[
\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} 4x^2 = 4(-2)^2 = 16
\]

(b) Compute \( \lim_{x \to -2^-} f(x) \)

Solution: Similarly, we have
\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} 3x - 2 = 3(-2) - 2 = -8
\]

(c) Is the function \( f \) continuous at \( x = -2 \)? Explain.

Solution: No, this function is not continuous at \( x = -2 \). This is because, as was shown in the two previous parts, that \( \lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x) \). This means that the two-sided limit does not exist. Therefore it is not true that \( \lim_{x \to -2} f(x) = f(-2) \), since the limit does not exist.

5. Let \( f(x) = x^3 - 3x^2 \). Given that \( f'(x) = 3x^2 - 6x \), find the equation of the line tangent to the graph of \( f \) at the point \( (x, y) \) corresponding to \( x = -2 \).

Solution: We use the point-slope form of the equation of a line: \( y - y_0 = m(x - x_0) \). So we need a point on the line and the slope of the line. We take \( x_0 = -2 \) and \( y_0 = f(-2) = (-2)^3 - 3(-2)^2 = -8 - 12 = -20 \). We also take the slope of the tangent line we are trying to construct to be \( m = f'(-2) = 3(-2)^2 - 6(-2) = 12 + 12 = 24 \).

To summarize: \( (x_0, y_0) = (-2, -20) \) and \( m = 24 \). Thus,

\[
\begin{align*}
    y - (-20) & = 24(x - (-2)) \\
    y + 20 & = 24(x + 2) \\
    y & = 24x + 48 - 20 \\
    \boxed{y & = 24x + 28}
\end{align*}
\]

6. Let \( y = f(x) \) be a function and let \( a \) be in Dom\( (f) \). Give the definition of the derivative of \( f \) at \( x = a \), which is denoted \( f'(a) \).

Solution:
\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

7. Let \( f(x) = \frac{4}{x} \). Compute the value of \( f'(x) \) from the definition of derivative.
Solution:

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} \]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{4}{x+h} - \frac{4}{x} \right) = \lim_{h \to 0} \frac{1}{h} \frac{4x - 4(x + h)}{x(x + h)} \]

\[ = \lim_{h \to 0} \frac{1}{h} \frac{-4h}{x(x + h)} = \lim_{h \to 0} \frac{-4}{x(x + h)} \]

\[ = -\frac{4}{x^2} \]

Thus,

\[ f'(x) = -\frac{4}{x^2} \]

8. Let \( f(x) = \sqrt{x} \). Compute \( f'(4) \) using the definition.

Solution: We have

\[ f(4) = 2 \]
\[ f(4 + h) = \sqrt{4 + h} \]
\[ f(4 + h) - f(4) = \sqrt{4 + h} - 2 \]
\[ \frac{f(4 + h) - f(4)}{h} = \frac{\sqrt{4 + h} - 2}{h} \]

\[ = \frac{\sqrt{4 + h} - 2}{h} \cdot \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2} \]
\[ = \frac{(4 + h) - 4}{h(\sqrt{4 + h} + 2)} \]
\[ = \frac{h}{h(\sqrt{4 + h} + 2)} \]
\[ = \frac{1}{\sqrt{4 + h} + 2} \]
9. A ball is thrown into the air with a velocity of 40 ft/s, its height in feet after \( t \) seconds is given by \( f(t) = 40t - 16t^2 \). Find the instantaneous velocity of the ball when \( t = 1 \).

Solution: Note that \( f(1) = 24 \).

\[
v(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}
= \lim_{h \to 0} \frac{40(1 + h) - 16(1 + h)^2 - 24}{h}
= \lim_{h \to 0} \frac{40 + 40h - 16 - 32h - 16h^2 - 24}{h}
= \lim_{h \to 0} \frac{8h - 16h^2}{h}
= \lim_{h \to 0} 8 - 16h
= 8
\]

The velocity of the ball after \( t = 1 \) second is \( v(1) = 8 \) ft/second.