1 Review 5

1.1 Characteristic polynomial

Definition 1 *The characteristic polynomial of an $n \times n$ matrix $A = (a_{ij})$ is defined by*

$$\det(A - \lambda I)$$

Example 2 *The characteristic polynomial of*

$$A = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

is

$$\begin{aligned} P(\lambda) &= \det\left(\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 7-\lambda & 2 \\ 6 & 3-\lambda \end{bmatrix}\right) \\ &= \lambda^2 - 10\lambda + 9 \end{aligned}$$

Note that

$$P(0) = 9 = \det\left(\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}\right)$$

In general, for an $n \times n$ matrix A the roots $\lambda_1, \ldots \lambda_n$ of $P(\lambda)$ are called the eigenvalues of the matrix A and satisfy

$$P(\lambda_j) = \det(A - \lambda_j I) = 0, j = 1, \ldots, n$$

Exercise 3 *Find the characteristic polynomial of*

$$A = \begin{bmatrix} 8 & 0 & -8 \\ -7 & -9 & 0 \\ 0 & 6 & 8 \end{bmatrix}$$

$$P(\lambda) = \det \begin{bmatrix} 8-\lambda & 0 & -8 \\ -7 & -9-\lambda & 0 \\ 0 & 6 & 8-\lambda \end{bmatrix} = -\lambda^3 + 7\lambda^2 + 80\lambda - 240$$

Next, determine the eigenvalues of A using two methods: First, find the roots of the characteristic polynomial, and second, using the eigenvalues command in the matrix submenu of the compute menu.

Exercise 4 *Find a matrix A that has characteristic polynomial $P(\lambda) = \lambda^2 - 4$ and $P(\lambda) = \lambda^4$.*

Note that many matrices have the same characteristic polynomial. To see this consider for $b \neq 0$

$$A = \left(\begin{bmatrix} a & b \\ \frac{1}{b} & d \end{bmatrix} \right)$$

which has characteristic polynomial

$$P(X) = X^2 + (-a - d)X + (ad - 1)$$

That is, the characteristic polynomial does not depend on b.

1.2 Square roots of matrices

Definition 5 *Some square matrices have square roots. That is, $\sqrt{A}$, if one exists, is a matrix B that satisfies*

$$B^2 = A$$

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

so $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a square root of itself. So is $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Clearly I_2 is a square root of itself. Interestingly, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is also a square root of I_2 because

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Likewise

$$\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

so $\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$ is a square root of $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$.

Exercise 6 *Find a square root of*

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Exercise 7 *Find a square root of*

$$\begin{matrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Exercise 8 *Let A be an $n \times n$ invertible matrix that has a square root. Is $\sqrt{A}$ invertible? If so, what is $\left(\sqrt{A}\right)^{-1}$. Hint: Consider the matrix $\left(\sqrt{A}A^{-1}\right)$.*

Exercise 9 *Let A be an $n \times n$ invertible matrix that has a square root. How are* $\det(A)$ *and* $\det\left(\sqrt{A}\right)$ *related?*

Exercise 10 *Not all matrices have square roots. Can you prove that* $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ *does not have a square root?*

1.3 Using formulas

Example 11 *Let*

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 7 & 1 \\ 1 & 4 & 6 \end{bmatrix}$$

and consider

$$\det(B) = 53$$

$$\det\left(B - \lambda \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) = 14\lambda^2 - 48\lambda - \lambda^3 + 53$$

Example 12 *Let*

$$F = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 3 & 4 & 5 \\ 4 & 2 & 2 & 0 \end{bmatrix}$$

and $\mathbf{b} = \begin{matrix} 2 \\ 3 \\ -1 \\ 4 \end{matrix}$. *Then*

$$F\mathbf{x} = \mathbf{b}$$

has solution

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} -\frac{1}{8} & \frac{1}{8} & 0 & \frac{3}{16} \\ \frac{7}{8} & \frac{1}{8} & 0 & -\frac{5}{16} \\ -\frac{5}{8} & -\frac{3}{8} & 0 & \frac{7}{16} \\ -\frac{1}{40} & \frac{9}{40} & \frac{1}{5} & -\frac{13}{80} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{7}{8} \\ \frac{7}{8} \\ -\frac{5}{8} \\ -\frac{9}{40} \end{pmatrix} \end{aligned}$$

Exercise 13 *Use the formula command to generate the characteristic polynomial, the inverse, and the transpose of a matrix C.*

Exercise 14 *Use the formula command to set up a multiple choice factoring question. (Three possible choices are enough. You may need to review the exam video or the SWP help.)*

Exercise 15 *Reproduce the following use of the formula command. Let* $f(x) = 2x + 3$*. Then* $f'(x) = 2$,

$$\int (2x+3)\,dx = 3x + x^2 + C$$

and its graph is given below x

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Instructions: Create a file containing the answers to the exercises in this lesson. You do not need to include the definitions and examples. Submit a .tex version of your file to teprice@uakron.edu. The name of your files should be of the form **yourlastnameR5.tex**. All calculations should be done using the CAS in SWP.