1 Lesson 17: Invertible matrices

Definition 1 *An $n \times n$ matrix A is **invertible** if there is a matrix B such that $AB = I_n$. The matrix B is called the inverse of A and is usually denoted by A^{-1}.*

Ex 2 *Let*

$$A = \begin{bmatrix} 824 & -65 & -814 & -741 \\ -979 & -764 & 216 & 663 \\ 880 & 916 & 617 & -535 \\ 597 & -245 & 79 & 747 \end{bmatrix}$$

Then

$$A^{-1} = \begin{bmatrix} \frac{355\,703\,801}{560\,204\,726\,312} & \frac{251\,945\,429}{560\,204\,726\,312} & \frac{333\,907\,521}{560\,204\,726\,312} & \frac{368\,376\,467}{560\,204\,726\,312} \\ -\frac{901\,700\,263}{560\,204\,726\,312} & -\frac{1431\,117\,995}{560\,204\,726\,312} & -\frac{674\,820\,423}{560\,204\,726\,312} & -\frac{107\,573\,749}{560\,204\,726\,312} \\ \frac{300\,825\,139}{560\,204\,726\,312} & \frac{1084\,283\,855}{560\,204\,726\,312} & \frac{925\,388\,595}{560\,204\,726\,312} & -\frac{1185\,903}{560\,204\,726\,312} \\ -\frac{13\,017\,657}{11\,919\,249\,496} & -\frac{16\,710\,637}{11\,919\,249\,496} & -\frac{12\,469\,153}{11\,919\,249\,496} & \frac{8944\,205}{11\,919\,249\,496} \end{bmatrix}$$

because

$$\begin{bmatrix} 824 & -65 & -814 & -741 \\ -979 & -764 & 216 & 663 \\ 880 & 916 & 617 & -535 \\ 597 & -245 & 79 & 747 \end{bmatrix} \begin{bmatrix} \frac{355\,703\,801}{560\,204\,726\,312} & \frac{251\,945\,429}{560\,204\,726\,312} & \frac{333\,907\,521}{560\,204\,726\,312} & \frac{368\,376\,467}{560\,204\,726\,312} \\ -\frac{901\,700\,263}{560\,204\,726\,312} & -\frac{1431\,117\,995}{560\,204\,726\,312} & -\frac{674\,820\,423}{560\,204\,726\,312} & -\frac{107\,573\,749}{560\,204\,726\,312} \\ \frac{300\,825\,139}{560\,204\,726\,312} & \frac{1084\,283\,855}{560\,204\,726\,312} & \frac{925\,388\,595}{560\,204\,726\,312} & -\frac{1185\,903}{560\,204\,726\,312} \\ -\frac{13\,017\,657}{11\,919\,249\,496} & -\frac{16\,710\,637}{11\,919\,249\,496} & -\frac{12\,469\,153}{11\,919\,249\,496} & \frac{8944\,205}{11\,919\,249\,496} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 3 *Let A be an $n \times n$ invertible matrix with inverse B. Then*

$$BA = I$$

That is, A is the inverse of B. Essentially, this means that A and its inverse commute ($AB = BA$). This also means that the inverse of a matrix is unique. To see this suppose B_1 and B_2 are inverses of an $n \times n$ matrix A. Then

$$AB_1 = I = AB_2$$

Hence, multiplying

$$AB_1 = AB_2$$

by B_1 we get

$$B_1AB_1 = B_1AB_2 \Longrightarrow IB_1 = IB_2 \Longrightarrow B_1 = B_2$$

Interestingly, there are algebraic structures for matrices for which a matrix A may have a right inverse which is different from its left inverse. For example, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then, as is easily seen by direct multiplication,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ r & q \end{bmatrix}$$

is a (right) inverse for A. Specifically,

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ r & q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, BA is a 3×3 matrix and moreover

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Theorem 4 *Let A and B be two $n \times n$ invertible matrices. Then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. *Observe that*

$$(AB)\left(B^{-1}A^{-1}\right) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

so that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. ■

Exercise 5 *Let A be an $n \times n$ invertible matrix and k a positive integer. Define $A^{-k} = \overbrace{A^{-1}A^{-1}\cdots A^{-1}}^{k\ copies}$. Prove that*

$$A^{-k} = \left(A^k\right)^{-1}$$

Exercise 6 *Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ so that $A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^{-1}$. By direct calculation prove that*

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \textit{iff} \quad A^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Exercise 7 *Suppose A is an $n \times n$ invertible matrix and $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ vectors. Prove that*

$$A^{-1}\mathbf{b}$$

is the only solution to the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

2 Project 17

Instructions: Create a file containing the answers to the exercises in this lesson. You do not need to include the definitions and examples. Submit a .tex version of your file to teprice@uakron.edu. The name of your files should be of the form **yourlastname17.tex**. All calculations should be done using the CAS in SWP.