

1 Lesson 15: More on matrices IV

1.1 Matrix multiplication

The product of an $m \times n$ matrix A with an $n \times p$ matrix B (note the dimensions) is the $m \times p$ matrix whose i, j -th entry is the inner product of the i -th row of A with the j -th column of B . If we use the notation $(AB)_{i,j}$ to denote the i, j -th element of the product of AB , then

$$(AB)_{i,j} = \text{row}(A, i) \text{col}(B, j) = \sum_{k=1}^n a_{i,k} b_{k,j}$$

for $i = 1, \dots, m$ and $j = 1, \dots, p$.

Example 1

$$\begin{aligned}
 & \begin{bmatrix} -4 & -1 \\ 5 & -7 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 4 & 0 & 2 \\ 7 & -5 & -3 & -1 \end{bmatrix} \\
 = & \left[\begin{array}{l} \text{row} \left(\begin{array}{cc} -4 & -1 \\ 5 & -7 \\ 1 & 0 \end{array}, 1 \right) \text{col} \left(\begin{array}{cccc} 8 & 4 & 0 & 2 \\ 7 & -5 & -3 & -1 \\ 1 & 1 \end{array}, 1 \right) \\ \text{row} \left(\begin{array}{cc} -4 & -1 \\ 5 & -7 \\ 1 & 0 \end{array}, 2 \right) \text{col} \left(\begin{array}{cccc} 8 & 4 & 0 & 2 \\ 7 & -5 & -3 & -1 \\ 1 & 1 \end{array}, 2 \right) \\ \text{row} \left(\begin{array}{cc} -4 & -1 \\ 5 & -7 \\ 1 & 0 \end{array}, 3 \right) \text{col} \left(\begin{array}{cccc} 8 & 4 & 0 & 2 \\ 7 & -5 & -3 & -1 \\ 1 & 1 \end{array}, 3 \right) \end{array} \right] \\
 = & \begin{bmatrix} -39 & -11 & 3 \\ & & 4 \end{bmatrix}
 \end{aligned}$$

Exercise 2 Fill in the remaining parts to the above example and then check your answer using the CAS in SWP.

Example 3 Note that

$$\begin{bmatrix} 8 & 4 & 0 & 2 & 7 \\ 7 & -5 & -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Exercise 4 In this problem you are to prove the general result suggested by the previous example. That is, let A be any $m \times n$ matrix and Z_n the $n \times 1$ zero vector. Prove that $AZ_n = Z_m$ where Z_m denotes the $m \times 1$ zero vector.

Note that both AB and BA are defined if and only if A and B are square matrices of the same size.

Exercise 5 Find two 2×2 matrices A and B such that $AB \neq BA$. That is matrix multiplication is **not** commutative. Find two 2×2 nontrivial matrices such that $AB = BA$

1.2 The transpose

Let $A = (a_{i,j})$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix $A^T = (a_{j,i})$. For, for example,

$$\begin{array}{cccccc}
 & & & & & 824 & -764 & 617 \\
 & & & & & 824 & -65 & -814 & -741 & -979 & ^T & -65 & 216 & -535 \\
 & & & & & -764 & 216 & 663 & 880 & 916 & = & -814 & 663 & 597 \\
 & & & & & 617 & -535 & 597 & -245 & 79 & & -741 & 880 & -245 \\
 & & & & & & & & & & & -979 & 916 & 79
 \end{array}$$

Note that the original matrix has dimension 3×5 while its transpose has dimension 5×3 .

Exercise 6 Let A be an $m \times n$ matrix. What is the dimension of AA^T ? $A^T A$?

Exercise 7 Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ and $B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$ be 2×2 matrices. Prove that

$$(AB)^T = B^T A^T$$

You may use the CAS of SWP to assist you with the multiplication but you should include some of the details. Of course, this result is true for the more general of two $n \times n$ matrices. Can you prove this? Hint: Just prove that $((AB)^T)_{i,j} = (B^T A^T)_{i,j}$.

Exercise 8 Prove that if A, B , and C are 2×2 matrices, then $(AB)C = A(BC)$. That is, matrix multiplication is associative.

2 Project 15

Instructions: Create a file containing the answers to the exercises in this lesson. You do not need to include the definitions and examples. Submit a .tex version of your file to teprice@uakron.edu. The name of your files should be of the form **yourlastname14.tex**. All calculations should be done using the CAS in SWP.