1 Lesson 12: More on matrices I

1.1 Definition of a matrix

A rectangular array of mathematical expressions is called a **matrix**. For example, the two by three (2×3) matrix (two rows and three columns)

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix}$$

is an array of real numbers. The entries (expressions) could be functions such as the following 4×2 matrix

$$\begin{bmatrix} \sin x & \cos x \\ e^x & \tan x \\ \cot x & \ln x \\ x^2+1 & \sec x \end{bmatrix}$$

Let m and n be positive integers. The general $m \times n$ matrix A is denoted by

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

where the $a_{i,j}$ are called the **entries** of A. The specific entry $a_{i,j}$ is called the i,j-th entry. If $m = 1$, then A is called an n-dimensional **row vector**. If $n = 1$ then A is an m-dimensional **column vector**. For example, $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ is 3-dimensional row vector while $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a 3-dimensional column vector.

Exercise 1 *Consider the matrix*

$$\begin{bmatrix} 824 & -65 & -814 & -741 & -979 & -764 & 216 & 663 \\ 880 & 916 & 617 & -535 & 597 & -245 & 79 & 747 \\ 477 & -535 & -906 & -905 & -266 & -8 & 765 & 448 \\ -348 & 470 & -608 & -686 & 702 & -61 & -49 & -433 \\ 966 & 902 & -942 & 712 & 761 & -892 & -564 & -826 \\ 251 & -414 & -44 & -214 & 235 & -781 & 421 & -340 \\ 881 & 444 & 360 & 932 & 700 & 725 & -750 & -637 \end{bmatrix}$$

What is the size of the above matrix? What is the $5,3$*-th entry?*

1.2 Generating a matrix in SWP

The process of generating a general matrix such as A above can be tedious. SWP can help.

1.2.1 Basics

Example 2 *Generate a 5×4 matrix containing all zeros.*

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Example 3 *Generate the 5×5 identity matrix*

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Exercise 4 *Generate a 4×5 matrix containing all fives. $f(i,j) = a_{i,j}$*

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{matrix}$$

Exercise 5 *Generate a $n \times n$ matrix containing all fives.*

Exercise 6 *Generate a 5×5 matrix with random entries.*

1.2.2 Defined functions can be used to generate a matrix.

Example 7 *The i,j-th entry of the Hilbert matrix is $\frac{1}{i+j-1}$. Define $g(i,j) = \frac{1}{i+j-1}$. Then*

$$\begin{matrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{matrix}$$

Example 8 *Let $f(i,j) = a_{i,j}$. Then*

Example 9 *Let* $g(i,j) = \begin{cases} 0 & if \quad i<j \\ 1 & if \quad i=j \\ 3 & if \quad i+j = 1 \operatorname{mod} 2 \\ 2 & if \quad i+j = 0 \operatorname{mod} 2 \end{cases}$

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 3 & 1 & 0 \\ 2 & 3 & 2 & 3 & 1 \\ 3 & 2 & 3 & 2 & 3 \\ 2 & 3 & 2 & 3 & 2 \end{matrix}$$

Exercise 10 *Use the Fill Matrix menu to generate the following matrix. Hint: Let* $h(i,j) = \begin{cases} 0 & if \quad i<j \\ 1 & if \quad ? \\ ? & if \quad ? \end{cases}$

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{matrix}$$

A matrix $(a_{i,j})$ for which $a_{i,j} = 0$ whenever $|i-j| > k$ is called a **band matrix** with band with $2k+1$. For example,

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

is a band matrix with band width 3.

Example 11 *Generate a* 6×6 *matrix with band width* 3.

$$\begin{matrix} b & c & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & 0 \\ 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & a & b \end{matrix}$$

Example 12 *Generate a* 10×10 *matrix with band width* 5.

$$\begin{matrix} c & d & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & c & d & e & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & d & e & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & c & d & e & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d & e & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c & d & e & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & d & e & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & c & d & e \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & c \end{matrix}$$

Exercise 13 *Use the Fill Matrix menu to generate*

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and the 17×17 *band matrix*

$$\begin{bmatrix} 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

2 Project 12

Instructions: Create a file containing the answers to the exercises in this lesson. You do not need to include the definitions and examples. Submit a .tex version of your file to teprice@uakron.edu. The name of your files should be of the form **yourlastname12.tex**. All calculations should be done using the CAS in SWP.