Pressureless Euler/Euler–Poisson systems via adhesion dynamics and scalar conservation laws *

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Abstract. The "sticky particles" model at the discrete level is employed to obtain global solutions for a class of systems of conservation laws among which lie the pressureless Euler and the pressureless attractive/repulsive Euler-Poisson system with zero background charge. We consider the case of finite, nonnegative initial Borel measures with finite second-order moment, along with continuous initial velocities of at most quadratic growth and finite energy. We prove the time regularity of the solution for the pressureless Euler system and obtain that the velocity satisfies the Oleinik entropy condition, which leads to a partial result on uniqueness. Our approach is motivated by earlier work of Brenier and Grenier who showed that one dimensional conservation laws with special initial conditions and fluxes are appropriate for studying the pressureless Euler system.

Key words. Presureless Euler, Euler-Poisson system, sticky particles, scalar conservation laws, Wasserstein distance, adhesion dynamics

AMS subject classifications. 35L65, 35L67, 82C40

1 Introduction

Let $\alpha,\ \beta \in I\!R$ and consider the system

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2) = \rho(\alpha\partial_x\Phi + \beta) \quad \text{in } I\!R \times (0,T). \\ \partial^2_{xx}\Phi = \rho \end{cases} \tag{1}$$

If $\alpha = \beta = 0$, then (1) describes the pressureless Euler system in spatial dimension one. The most commonly known form of the pressureless, attractive/repulsive Euler-Poisson system with zero background charge is also obtained from (1) by taking $\alpha = \pm 1$ and $\beta = 0$. In this paper, we are concerned with global existence of solutions for the initial value problem. Unlike the Euler with pressure case, the natural environment for the evolution is the space of nonnegative Borel measures on the real line. We consider the case of finite total mass, which we normalize

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to unity. The pressureless Euler ($\alpha = \beta = 0$) problem was studied by different techniques in [5], [6], [7], [9], [12], [16], [17], [18], [19], [21] etc. We point out that in these papers, generally, the velocity is taken to be at least bounded on the support of the initial measure. It appears that [12] and [21] are the only references (to our knowledge) which allow for unbounded velocities. Also, [12] is remarkable for dealing with the gravitational term as well ($\alpha = -1,\ \beta = 0$). In spite of that, the serious limitation of [12] is the assumptions that the initial velocity be sublinear growth and the initial mass distribution ρ_0 be either discrete or absolutely continuous with respect to the Lebesgue measure. Our main contribution is proving existence of global solutions for (1) if ρ_0 is just in $\mathcal{P}_2(I\!\!R)$, and v_0 is continuous of at most quadratic growth and finite energy. As a consequence of an important result from [14], we also manage to show that the solution we obtain for Euler pressureless satisfies the Oleinik entropy condition which was conjectured in [7] and [12]. Note that similar constraints to ours on ρ_0 and v_0 were anticipated by Shnirelman [21] for the pressureless Euler system. However, our approach is more general than that of Shnirelman which cannot easily accommodate the α and β terms in the right hand side of the momentum equation. The solutions constructed by Shnirelman were expressed in the form of a variational problem, which was shown in [3] to be equivalent to the variational principle considered in [12].

During the last decade, significant progress has been achieved in the study of partial differential equations in the context of optimal mass transportation. Much of the work on parabolic, dissipative equations was synthetized and placed in a very general setting in [2]. Much more recent and much less explored is the study of Hamiltonian systems in this context [1], [15]. The connection with the pressureless Euler system in arbitrary dimension was discovered by Benamou and Brenier [4] who showed that this system describes the geodesics in the Wasserstein space $\mathcal{P}_2(I\!\!R)$. By definition, $\mathcal{P}_p(I\!\!R)$ is the set of all Borel probabilities on $I\!\!R$ with finite p–order moment. The set $\mathcal{P}_2(I\!\!R)$ is endowed with the quadratic Wasserstein metric defined by

$$W_2^2(\mu,\nu) := \min_{\gamma} \int_{I\!\!R^2} |x-y|^2 d\gamma(x,y),$$

where the infimum is taken among all probabilities γ on the the product space $I\!\!R^2$ with marginals $\mu,\ \nu$. The theory of absolutely continuous curves in $\mathcal{P}_2(I\!\!R)$ [2] asserts the existence of velocities satisfying the conservation of mass equation in (1), regarded as a continuity equation. The left hand side of the momentum equation can also be interpreted as the acceleration along the curve. We shall discuss these interesting connections at the end of this paper. A different version of the pressureless Euler-Poisson system was analyzed in [14] in the context of optimal mass transportation. The focus was on the two-point boundary problem, and existence and uniqueness for solutions as action-minimizing paths in $\mathcal{P}_2(I\!\!R)$ was obtained.

We shall need the following assumptions:

$(H1)$ *The initial distribution of mass* $\rho_0 \in \mathcal{P}_2(I\!\!R)$;

$(H2)$ *There exists* $0 \le \Lambda < +\infty$ *such that*

$$v_0 \in C(I\!\!R) \cap L^2(\rho_0) \text{ and } |v_0(x)| \le \Lambda(1+x^2) \text{ for all } x \in I\!\!R.$$

The main objective is the following result.

Theorem 1.1. *The initial-value problem for (1) admits a global weak solution in the sense of distributions if* $(H1)$, $(H2)$ *hold.*

Two independent papers that appeared in 1996 and 1998 used adhesion dynamics to obtain global solutions for (1) in the $\alpha = \beta = 0$ case [7], [12] and in the $\alpha = -1$, $\beta = 0$ case [12]. Not only are we able to deal with the more general (1), but we also establish our results under less restrictive conditions on the initial distribution and velocity. In [7] the initial ρ_0 is compactly supported, while the initial velocity v_0 is continuous and bounded. These assumptions are relaxed in [12], e.g. $\sup_{|x|\leq R} |v_0(x)|/R \to 0$ as $R \to +\infty$ while $\mathrm{spt}(\rho_0)$ may be unbounded, in which case $\int_0^x y \mathrm{d}\rho_0(y) \to +\infty$ as $|x| \to +\infty$. As opposed to [7], however, [12] makes the extra-assumptions that ρ_0 be either discrete or absolutely continuous with respect to the Lebesgue measure, in which case $\rho_0 > 0$ on $\mathrm{spt}(\rho_0)$.

More recent work [17] treats the case $\alpha = \beta = 0$ for nonnegative Radon measures ρ_0 (not necessarily of finite total mass) and velocities $v_0 \in L^\infty(\rho_0)$. This paper is also remarkable in that it gives a necessary and sufficient condition for uniqueness of the Oleinik entropy solution: the initial weak continuity of the energy. As the example showing the necessity of this condition involves infinite mass initial measures, it remains unclear whether that is really needed in the finite mass case. Another paper that deals with possibly discontinuous (but bounded) initial velocities is [9], where the solution is produced constructively.

In [5] we find the concept of duality solutions, based on earlier work by same authors. Existence and uniqueness are obtained under the assumption of atom-free initial density and bounded and continuous initial velocity. Boudin [6] obtains global existence of smooth solutions when the initial data has some higher regularity and is bounded away from zero and infinity (thus, of infinite total mass). Interestingly, the initial velocity does not have to be nondecreasing in order to rule out formation of singularities in finite time. We will consider this issue in Section 4.3.

Whereas [7] and [12] use different approaches, they are still closely connected in principle. The fundamental underlying assumption for the discrete dynamics is the "sticky particle" hypothesis. The idea goes back to Zeldovich [24] and can be briefly described as follows. If $m_i,\ i = \overline{1,n}$ is a discrete system of masses initially located at $-\infty < x_1 < ... < x_n < +\infty$ and moving with initial velocities $v_i,\ i = \overline{1,n}$, then one makes the assumption that the velocities remain constant while there is no collision. At the collision of a group of particles, the particles stick together and the initial velocity of the newly formed particle is given by the conservation of momentum. In [12] the authors successfully implemented a version adapted to the case $\alpha = -1,\ \beta = 0$. Instead of the constant speed inter-collisional motion, we now assume uniformly accelerated motion between collisions. The acceleration is of gravitational nature and is proportional to the difference between the total mass to the left and the total mass to the right. Thus, in both cases, the trajectory of the i^{th} particle before collision is given by

$$x_i(t) = x_i + tv_i + \frac{t^2}{2} a_n^i,$$

where

$$a_n^i = \begin{cases} 0 & \text{if } \alpha = \beta = 0, \\ \frac{1}{2}\left(\sum_{j<i} m_j - \sum_{j>i} m_j\right) & \text{if } \alpha = -1,\ \beta = 0. \end{cases}$$

(Here we convene that $m_0 = m_{n+1} = 0$.) If the masses $m_j,\ i \le j \le k$ collide at time $t_0 > 0$, then conservation of momentum yields

$$v_i(t_0+) = \frac{\sum_{j=i}^k m_j v_j(t_0-)}{\sum_{j=i}^k m_j} \ .$$

Of course, only finitely many collisions can occur, therefore, the evolution of the system is completely determined by the above assumptions.

Next we briefly describe the technique employed in [12], whose approach does not distinguish between the discrete and absolutely continuous case. Here the problem is attacked from a "continuation of characteristics" point of view. When shocks occur, i.e. when the map $\phi_t(y) := y + tv_0(y) + t^2 a_0(y)/2$ is no longer invertible, one needs to redefine $\phi_t(y)$ in such a way that it remains nondecreasing and $\phi_{t\#}\rho_0 =: \rho_t$ satisfies the equation in a weak sense. This redefinition uses the so called Generalized Variational Principle which comes from the intuition provided by the discrete case (see [12] for details).

A more elegant approach [7], in our opinion, makes use of standard results on approximations for scalar conservation laws. It is applied to the Euler pressureless system ($\alpha = \beta = 0$) and relies on the fact that the distribution function M of ρ satisfies an autonomous scalar conservation law. Another advantage lies in the fact that the solution for the continuous problem is obtained from the discrete ones via approximation theory for scalar conservation laws. We shall adopt this point of view and prove the more general Theorem 1.1 by an appropriate adaptation of Brenier and Grenier's method.

The plan is as follows: we use $(H1)$ to produce a sequence of discrete probabilities

$$\rho_0^n := \sum_{i=1}^n m_i^{(n)} \delta_{x_i^{(n)}} \to \rho_0 \text{ as } n \to \infty$$

in the 2–Wasserstein distance. We shall, in fact, prove that this sequence may be taken such that $\int_{I\!R} \zeta \mathrm{d}\rho_0^n$ is uniformly bounded for some super-quadratic growth function $\zeta : [0,\infty) \to [0,\infty)$. Denote by M_0^n the right continuous distribution function of ρ_0^n and let

$$a_n(m) := \alpha\left(\sum_{j=1}^i m_j - \frac{1}{2} m_i\right) + \beta \tag{2}$$

whenever $M_0^n\big(x_i^{(n)}-\big) \le m < M_0^n\big(x_i^{(n)}\big)$. We then define flux functions $\tilde F_n : [0,T] \times [0,1] \to I\!R$ as follows

$$\tilde F_n(t,m) := \int_0^m f_n(\omega)\mathrm{d}\omega + t\int_0^m a_n(\omega)\mathrm{d}\omega \ , \tag{3}$$

where $N_{0\ \#}^n \chi_{(0,1)} = \rho_0^n$ optimally and $f_n := v_0 \circ N_0^n$ (the map N_0^n is taken to be right-continuous). By adhesion dynamics we construct the unique entropy solution M_n for the first-order problem

$$\partial_t M + \partial_x\big[\tilde F_n(t,M)\big] = 0, \quad M(0,\cdot) = M_0^n \ . \tag{4}$$

We then use $(H2)$ to show that, for an appropriate choice of approximating initial data, the sequence M_n will converge in some sense to the unique entropy solution for

$$\partial_t M + \partial_x\big[\tilde F(t,M)\big] = 0, \quad M(0,\cdot) = M_0 \ , \tag{5}$$

where $\tilde{F} : [0,T] \times [0,1] \to I\!R$ is given by

$$\tilde{F}(t,m) := \int_0^m f(\omega)\mathrm{d}\omega + t\int_0^m a(\omega)\mathrm{d}\omega = \int_0^m f(\omega)\mathrm{d}\omega + t(\alpha m^2/2 + \beta m)\ , \tag{6}$$

for $f := v_0 \circ N_0$ and $a(m) := \alpha m + \beta$ for $m \in [0,1]$. Here $N_0 := M_0^{-1}$ (generalized inverse) [13] is the right-continuous optimal map pushing $\chi_{(0,1)}$ forward to ρ_0. The solution M of (5) will produce the solution $\rho := \partial_x M$ and $v\rho := \partial_x[\tilde{F}(t,M)]$ for (1) via a generalization of a result due to Volpert [23] on BV calculus. The last section is dedicated to the $\alpha = 0 = \beta$ case. We give an explicit formula for v in terms of M and, more importantly, we prove that our solution satisfies the Oleinik entropy condition. Some qualitative properties of the solution are discussed, e.g. time regularity, and the impact of the initial velocity on the occurrence of spatial singularities. We finish with a partial result on uniqueness, i.e. we show that the energy of our solution for the pressureless Euler system is weakly continuous initially, which, along with the Oleinik entropy condition, leads to uniqueness in the case of bounded initial velocities [17].

2 Elements of one-dimensional BV calculus

2.1 A BV chain rule in dimension one

To prove a chain rule for BV functions, we need the following lemma:

Lemma 2.1. *Let μ be a Borel probability measure on $I\!R$ and M be its right-continuous distribution function. Write*

$$\mu = \sum_{j\in J} m_i \delta_{x_j} + \rho,$$

where $\{x_j\}_{j\in J}$ is the set (at most countable) of discontinuities of M, and $m_j := \mu(\{x_j\})$. If ρ is nonzero, then we have

$$M_{\#}\rho = \chi_{U^c} \text{ for } U := \bigcup_{j\in J} \big(M(x_j-), M(x_j)\big). \tag{7}$$

Proof: We first observe that the balance of mass is satisfied. Since M is monotone nondecreasing, (7) is equivalent to the fact that M is the optimal map pushing forward ρ to χ_{U^c}. That is what we prove next. It is well-known [13] that, since both measures are atom-free, the optimal map is given by $G^{-1} \circ F$, where F, G are the right-continuous distributions functions of ρ and χ_{U^c} respectively, and G^{-1} is the generalized inverse of G, given by $G^{-1}(y) = \inf\{m \in [0,1] : G(m) > y\}$. We shall show that $M(x) = G^{-1} \circ F(x)$ for ρ–a.e. $x \in \mathrm{spt}(\rho)$, i.e. $G^{-1}(F(x)) = \inf\{m \in [0,1] : G(m) > F(x)\} = M(x)$. Note that

$$F(x) = M(x) - \sum_{x_j \le x} m_j, \text{ and } G(m) = m - \sum_{M(x_j)\le m} m_j \text{ if } m \in U^c.$$

Thus, if $m \in U^c$,

$$G(m) - F(x) = \begin{cases} m - M(x) - \sum_{M(x)<M(x_j)\le m} m_j & \text{if } m > M(x), \\ 0 & \text{if } m = M(x), \\ m - M(x) + \sum_{m<M(x_j)\le M(x)} m_j & \text{if } m < M(x). \end{cases}$$

Since $G(M(x)) - F(x) = 0$ and G is a right-continuous, nondecreasing function, all we need to prove is that there does not exist a nondegenerate interval $[M(x), M(x)+\epsilon]$ such that $G(m) = F(x)$ for all $m \in [M(x), M(x)+\epsilon]$. Suppose such an interval exists. We know M is right-continuous at x, therefore if $x < z < x + \delta(\epsilon)$ we have that $M(x) \leq M(z) \leq M(x) + \epsilon$. Thus, as $M(z) \in U^c$, we infer $M(z) - M(x) - \sum_{x<x_j\leq z} m_j = 0$, i.e. $\rho([x,z]) = 0$. This means $x \in \partial\mathrm{spt}(\rho)$, and, by taking $[x, z_x]$ the maximal interval for which $\rho([x,z]) = 0$, we see that $(x, z_x) \cap (y, z_y) = \emptyset$ for any $x \neq y$ in $\partial\mathrm{spt}(\rho)$ for which these nondegenerate intervals exist. This means that there are at most countably many such points and, since ρ is atom-free, it follows that the ρ–measure of this set of points is zero. The lemma is thus proved. QED.

The following theorem is fundamental. The first part of (i) can be trivially obtained from [23], see, e.g. [5] for the exact formula on the derivative. The novelty of our result appears in (ii), as we go from the Lipschitz to the $W^{1,p}$ case. However, for the reader's convenience we sketch an elementary proof for (i) as well.

Theorem 2.2. *Let μ be a Borel probability measure on $I\!R$ and let M be its right-continuous distribution function.*
(i) Assume $f \in W^{1,\infty}(0,1)$. Then, $f \circ M \in BV(I\!R)$ with distributional derivative $g\mu$, where

$$g(x) := \begin{cases} \overline{f' \circ M}(x) & \text{if } \mu(\{x\}) = 0, \\ \dfrac{f \circ M(x) - f \circ M(x-)}{\mu(\{x\})} & \text{if } \mu(\{x\}) \neq 0, \end{cases} \tag{8}$$

μ–a.e. for some function $\overline{f' \circ M} \in L^\infty(I\!R)$. Furthermore, suppose f' is defined unambiguously on $(0,1)$, and there exists a bounded $C^1[0,1]$ sequence f_n such that $f_n \to f$ uniformly and $f_n' \to f'$ everywhere *on $(0,1)$. Then $\overline{f' \circ M} \equiv f' \circ M$ in the μ–a.e. sense, or equivalently,*

$$g(x) = \int_0^1 f'\big((1-s)M(x-) + sM(x)\big)\mathrm{d}s, \text{ for } \mu\text{–a.e. } x \in I\!R. \tag{9}$$

(ii) Let $1 \leq p < +\infty$ and assume $f \in W^{1,p}(0,1) \cap C^1(0,1)$ and g is given by (8), with $\overline{f' \circ M} \equiv f' \circ M$ in the μ–a.e. sense. Then $g \in L^p(\mu)$ with $\|g\|_{L^p(\mu)} \leq \|f'\|_{L^p(0,1)}$ and $f \circ M \in BV(I\!R)$ with distributional derivative $g\mu$.
Now take $f \in W^{1,p}(0,1)$. Suppose f' is defined unambiguously on $(0,1)$, and there exists a $W^{1,p}(0,1) \cap C^1(0,1)$ sequence f_n such that $f_n \to f$ in $W^{1,p}$ and $f_n' \to f'$ everywhere *on $(0,1)$. Then we still have the same result with $\overline{f' \circ M} \equiv f' \circ M$ in the μ–a.e. sense.*

Proof: W.l.o.g. we may assume that μ is supported in some bounded interval I. Also, we shall first assume $f \in C^1[0,1]$ to show (8) is valid with $\overline{f' \circ M} \equiv f' \circ M$. Now consider $\varphi \in C_c^\infty(I)$. We need to show that

$$-\int_I \varphi'(x) f \circ M(x)\mathrm{d}x = \int_I \varphi(x) g(x)\mathrm{d}\mu(x). \tag{10}$$

If μ has finitely many atoms, then the validity of (10) can be checked by direct computation. Indeed, since M is piecewise continuous and bounded, we may approximate it uniformly by nondecreasing piecewise $W^{1,1}$ functions M_ϵ (may take M_ϵ piecewise linear and continuous on each continuity interval for M, such that M and M_ϵ agree at the endpoints). Thus, $\mu_\epsilon := M_\epsilon' \rightharpoonup$

$M' = \mu$ weakly as nonnegative, bounded measures. The chain rule for Sobolev functions [8] applies piecewise and yields (10) for μ_ϵ. Then we pass to the limit to obtain the result for μ. Thus, let us assume $D := \{x_1, x_2, ..., x_n, ...\}$ is the infinite set of all atoms of μ and write

$$\mu = \sum_{i=1}^{\infty} m_i \delta_{x_i} + \rho, \text{ where } m_i > 0, \ i = 1, 2, ...$$

and ρ is an atom-free nonnegative Borel measure of total mass $1 - \sum_{i=1}^{\infty} m_i \geq 0$. We shall call the atomic measure the singular part while ρ shall be called the regular part (although it may not be absolutely continuous with respect to the Lebesgue measure). Consider now the sequence of measures μ_n given by $\mu_n = \sum_{i=1}^{n} m_i \delta_{x_i} + \rho$. Of course, $\mu_n \rightharpoonup \mu$ weakly $\star$ as measures. Since M_n, the right-continuous distribution function of μ_n, has only finitely many discontinuities, (8) holds for μ_n as proved above. It is easy to see that $M_n \to M$ Lebesgue a.e., thus, the continuity of f along with its boundedness gives the convergence of the left hand side of (10) by dominated convergence. Therefore, $f \circ M_n \in BV(I\!R)$ with distributional derivative $g_n \mu_n$, where

$$g_n(x) := \begin{cases} f' \circ M_n(x) & \text{if } \mu_n(\{x\}) = 0, \\ \dfrac{f \circ M_n(x) - f \circ M_n(x-)}{\mu_n(\{x\})} & \text{if } \mu_n(\{x\}) \neq 0. \end{cases}$$

Now we write

$$\int_{I\!R} \varphi(x) g_n(x) \mathrm{d}\mu_n(x) = \sum_{i=1}^{n} \varphi(x_i)[f(M_n(x_i)) - f(M_n(x_i-))] + \int_{I\!R} \varphi(x) f' \circ M_n(x) \mathrm{d}\rho(x).$$

By the continuity of f' we obtain the convergence of the second term in the right hand side. Thus, it remains to prove that

$$\sum_{i=1}^{n} \varphi(x_i)[f(M_n(x_i)) - f(M_n(x_i-))] \to \sum_{i=1}^{\infty} \varphi(x_i)[f(M(x_i)) - f(M(x_i-))] \ ,$$

which can be obtained after some calculations as a consequence of the convergence of the series $\sum m_i$ if $f \in C^2[0,1]$, then for $C^1[0,1]$ functions by approximation. If f is $W^{1,\infty}(0,1)$, then we conclude by taking a sequence of $C^1[0,1]$–functions such that $f_n \to f$ uniformly and f_n' are uniformly bounded. Indeed, one has

$$-\int_{I\!R} \varphi' f_n \circ M \mathrm{d}x = \int_{I\!R} \varphi f_n' \circ M \mathrm{d}\rho + \int_{I\!R} \varphi(x) \frac{f_n(M(x)) - f_n(M(x-))}{M(x) - M(x-)} \mathrm{d}\mu_s(x),$$

where $\mu_s = \sum m_j \delta_{x_j}$ is the singular part of μ. Note that the ratio in the second term of the right hand side converges uniformly to $[f(M(x)) - f(M(x-))]/[M(x) - M(x-)]$ on the support of μ_s, thus the uniform bound on f_n' ensures the convergence of the integral by dominated convergence. Since the left hand side is trivially convergent, it follows that

$$\int_{I\!R} \varphi(x) f_n' \circ M(x) \mathrm{d}\rho(x) \text{ converges as } n \to \infty,$$

which, along with the uniform bound on f_n', yields the convergence of $f_n' \circ M$ in the L^∞ weak $\star$ topology. We also deduce that the limit, denoted by $\overline{f' \circ M}$, is μ–a.e. independent of the

chosen sequence f_n. The second statement from (i) easily follows (by dominated convergence) from the fact that $g_n \to g$ everywhere as an L^∞ bounded sequence.
To prove the first part of (ii) we truncate f' by

$$f_n'(x) := \begin{cases} -n & \text{if } f'(x) < -n, \\ f'(x) & \text{if } |f'(x)| \le n, \\ n & \text{if } f'(x) > n \end{cases} \tag{11}$$

and let f_n be the antiderivative of f_n' vanishing at zero. Note that $|f_n'| \le |f'|$ on $(0,1)$, which implies f_n is uniformly bounded with respect to n in $L^p(0,1)$. First, since $f_n \in C^1[0,1]$, we infer, according to (i), that $f_n \circ M$ is BV with distributional derivative $g_n\mu$, where

$$g_n(x) := \begin{cases} f_n' \circ M(x) & \text{if } \mu(\{x\}) = 0, \\ \dfrac{f_n \circ M(x) - f_n \circ M(x-)}{\mu(\{x\})} & \text{if } \mu(\{x\}) \ne 0. \end{cases} \tag{12}$$

This (see (9) for the equivalent integral expression) together with the fact $|f_n'| \le |f'|$ implies

$$|g_n(x)| \le \int_0^1 \big|f'\big((1-s)M(x-) + sM(x)\big)\big| \mathrm{d}s =: h(x), \text{ for } x \in I.$$

Since $f_n' \to f'$, we infer $g_n \to g$ pointwise. Assume that $h \in L^p(\mu)$. Then, we may pass to the limit in the right hand side of

$$-\int_I \varphi'(x) f_n \circ M(x) \mathrm{d}x = \int_I \varphi(x) g_n(x) \mathrm{d}\mu(x).$$

The left hand side converges to the appropriate quantity because $f_n \to f$ uniformly and f_n are uniformly bounded.
Thus, we are done if we can prove that $h \in L^p(\mu)$. For this let $\{x_j\}_{j\in J}$ be the set of discontinuities of M and $m_j := \mu(\{x_j\})$. Now consider, for $s \in [0,1]$, the sum

$$\sum_{i=1}^n m_i \big|f'\big((1-s)M(x_i-) + sM(x_i)\big)\big|^p = \sum_{i=1}^n m_i \big|f'\big(M(x_i-) + sm_i\big)\big|^p.$$

By monotone convergence

$$\sum_{i=1}^n \int_0^1 m_i \big|f'\big(M(x_i-) + sm_i\big)\big|^p \mathrm{d}s \to \int_0^1 \sum_{i=1}^\infty m_i \big|f'\big(M(x_i-) + sm_i\big)\big|^p \mathrm{d}s$$

which, after obvious linear changes of variables, is equivalent to

$$\int_{\cup_{i=1}^n [M(x_i-), M(x_i)]} |f'(m)|^p \mathrm{d}m \to \int_0^1 \sum_{i=1}^\infty m_i \big|f'\big(M(x_i-) + sm_i\big)\big|^p \mathrm{d}s.$$

Again, by monotone convergence the left hand side converges to $\int_U |f'(m)|^p \mathrm{d}m$, where $U := \bigcup_{j\in J} \big(M(x_j-), M(x_j)\big)$. Thus, since $f' \in L^p(0,1)$ and $U \subset [0,1]$, it follows that

$$\int_0^1 \sum_{i=1}^\infty m_i \big|f'\big(M(x_i-) + sm_i\big)\big|^p \mathrm{d}s = \int_U |f'(m)|^p \mathrm{d}m \le \|f'\|^p_{L^p(0,1)},$$

so we can apply Fubini's Theorem to obtain

$$\int_I \int_0^1 \left|f'\big((1-s)M(x-)+sM(x)\big)\right|^p \mathrm{d}s\mathrm{d}\mu_s(x) = \int_U |f'(m)|^p \mathrm{d}m < +\infty, \tag{13}$$

where μ_s denotes, as before, the singular part of μ (unrelated to the integration variable s). If $\mu = \mu_s$, then we are done. Else, by using Lemma 2.1 we obtain

$$\int_I |f' \circ M(x)|^p \mathrm{d}\rho(x) = \int_{U^c} |f'(m)|^p \mathrm{d}m,$$

which, combined with (13), yields

$$\int_I \int_0^1 \left|f'\big((1-s)M(x-)+sM(x)\big)\right|^p \mathrm{d}s\mathrm{d}\mu(x) = \|f'\|^p_{L^p(0,1)}. \tag{14}$$

Therefore, $h \in L^p(\mu)$ with $\|h\|_{L^p(\mu)} \le \|f'\|_{L^p(0,1)}$ (with equality if $p = 1$). Since $|g| \le h$ pointwise, the proof of the first part of (ii) is concluded.
Consequently,

$$-\int_{I\!R} \varphi' f_n \circ M \mathrm{d}x = \int_{I\!R} \varphi g_n \mathrm{d}\mu \text{ for all positive integers } n\ , \tag{15}$$

where g_n is defined by (12) for the approximating sequence f_n considered in the second part of (ii). We have, just as in deducing (14), that

$$\begin{aligned}\int_I |g_n(x) - g_m(x)|^p \mathrm{d}\mu(x) \ &\le \ \int_I \int_0^1 |(f'_n - f'_m)\big((1-s)M(x-)+sM(x)\big)|^p \mathrm{d}s\mathrm{d}\mu(x) \\ &= \ \|f'_n - f'_m\|^p_{L^p(0,1)}\end{aligned}$$

for all natural $m,\ n \ge 1$. Thus, $\{g_n\}$ is convergent in $L^p(\mu)$ and, due to the hypothesis of *everywhere* convergence of f'_n to f', we obtain $g_n \to g$ in $L^p(\mu)$, where

$$g(x) := \int_0^1 f'\big((1-s)M(x-)+sM(x)\big)\mathrm{d}s\ .$$

Passing to the limit in (15) concludes our proof. QED.

The following corollary holds for any $1 \le p \le +\infty$.

Corollary 2.3. *Let $f \in L^p(0,1)$ be right-continuous (thus, unambiguously defined everywhere in $(0,1)$) and take F to be its antiderivative vanishing at zero. If M is the cumulative distribution function of some Borel probability measure μ on $I\!R$, then $F \circ M \in BV(I\!R)$ with distributional derivative $g\mu$, where*

$$g(x) := \begin{cases} f \circ M(x) & \text{if } \mu(\{x\}) = 0, \\ \dfrac{F \circ M(x) - F \circ M(x-)}{\mu(\{x\})} & \text{if } \mu(\{x\}) \neq 0, \end{cases} \tag{16}$$

in the μ-a.e. sense. Furthermore, $g \in L^p(\mu)$ and $\|g\|_{L^p(\mu)} \le \|f\|_{L^p(0,1)}$.

The proof is an immediate consequence of Theorem 2.2. Indeed, we take (upon extending f by zero outside $(0,1)$) the function $f_n := \eta^n * f$, where η^n is obtained from the standard mollifier supported in $[-1/n, 1/n]$ by shifting it to the left by $1/n$. The classic properties of mollification still hold, e.g. $f_n \to f$ in $L^p(0,1)$ (if $p \neq +\infty$) and $f_n \in C^\infty[0,1]$. However, the interesting feature of these "shifted mollifiers" is that the right-continuity of f on $(0,1)$ is enough to easily prove that $f_n \to f$ *everywhere* in $(0,1)$. Thus, if we take $F_n(m) := \int_0^m f_n(\omega)\mathrm{d}\omega$, we are within the hypotheses of Theorem 2.2.

Remark 2.4. *To understand the relevance of this result, observe that the functions f_n, f defined in Introduction are right-continuous on $(0,1)$. Indeed, that comes as a consequence of the continuity of v_0 on $I\!R$ and the right-continuity of N_0^n, N_0 on $(0,1)$ (as generalized inverses of nondecreasing, right-continuous functions [22]).*

2.2 A two-dimensional extension

Now let us assume $M : [0,T] \times I\!R \to [0,1]$ for some $0 < T < +\infty$, such that $M \in BV([0,T] \times I\!R)$ and $M(t,\cdot)$ is a right-continuous probability distribution function for Lebesgue a.e. $t \in (0,T)$. The following lemma is easy to check as a consequence of Fubini's Theorem.

Lemma 2.5. *Let ∇M be the vector-valued measure given by the BV gradient $\nabla = (\partial_t, \partial_x)$ of M. Then, its x–component $\mathrm{d}\partial_x M$ admits the decomposition $\mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t$.*

We shall use this to prove the main result of this section, result that we state below. We denote by C_r the space of right-continuous functions.

Theorem 2.6. *Consider $M : [0,T] \times I\!R \to [0,1]$ as above, let $f \in L^2(0,1) \cap C_r(0,1)$ and F be its antiderivative vanishing at zero. Assume further that*

$$\partial_t M + \partial_x [F \circ M] = 0 \ \textit{in}\ \mathcal{D}'([0,T] \times I\!R). \tag{17}$$

Then $F \circ M \in BV([0,T] \times I\!R)$ and $\nabla[F \circ M] = g \nabla M$ for some $g \in L^1(|\nabla M|)$.

Proof: We may consider, w.l.o.g., the case $f \in C(0,1)$. Else, we simply use Corollary 2.3 instead of Theorem 2.2. Let us start by taking the truncations f_n for f as in (11). The corresponding F_n's are in $C^1[0,1]$, $F_n \to F$ uniformly and $f_n \to f$ in $L^2(0,1)$. According to [23], $F_n \circ M \in BV([0,T] \times I\!R)$ and there exist $\bar{f}_n \in L^\infty$ such that $\nabla[F_n \circ M] = \bar{f}_n \nabla M$ as vector-valued measures. This means

$$-\int_0^T \int_{I\!R} F_n(M) \nabla \cdot \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t = \int_0^T \int_{I\!R} \bar{f}_n \phi_1 \mathrm{d}\partial_t M + \int_0^T \int_{I\!R} \bar{f}_n \phi_2 \mathrm{d}\partial_x M, \tag{18}$$

for all $\boldsymbol{\phi} := (\phi_1, \phi_2) \in C_c^\infty((0,T) \times I\!R; I\!R^2)$. We now use Lemma 2.5 and (17) to infer

$$-\int_0^T \int_{I\!R} F_n(M) \nabla \cdot \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t = \int_0^T \int_{I\!R} \big(-g\phi_1 + \phi_2\big) \bar{f}_n \mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t, \tag{19}$$

where, according to Theorem 2.2 (*ii*)

$$g(t,\cdot) := \int_0^1 f\big((1-s)M(t,\cdot-) + sM(t,\cdot)\big)\mathrm{d}s \in L^2(\partial_x M(t,\cdot))$$

for Lebesgue a.e. $t \in (0,T)$. Also, Theorem 2.2 (ii) ensures

$$\|g(t,\cdot)\|_{L^2(\partial_x M(t,\cdot))} \le \|f\|_{L^2(0,1)} \text{ uniformly with respect to } t.$$

Note that (19) implies

$$-\int_0^T\int_{I\!R} F_n(M)\partial_x\phi_1 \mathrm{d}x\mathrm{d}t = \int_0^T\int_{I\!R} \bar{f}_n\phi_1 \mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t, \tag{20}$$

for all $\phi_1 \in C_c^\infty((0,T)\times I\!R)$. For Lebesgue a.e. $t \in (0,T)$ one has (Theorem 2.2 (i))

$$\partial_x[F_n \circ M(t,\cdot)] = g_n(t,\cdot)\partial_x M(t,\cdot) \text{ for } g_n(t,x) := \int_0^1 f_n\big((1-s)M(t,x-) + sM(t,x)\big)\mathrm{d}s$$

and

$$\|g_n(t,\cdot)\|_{L^2(\partial_x M(t,\cdot))} \le \|f\|_{L^2(0,1)} \text{ uniformly in } t \text{ and } n.$$

Along with (20), this implies $\bar{f}_n \equiv g_n$ in the $\mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t$–a.e. sense. Therefore, (19) is equivalent to

$$-\int_0^T\int_{I\!R} F_n(M)\nabla\cdot\boldsymbol{\phi}\mathrm{d}x\mathrm{d}t = \int_0^T\int_{I\!R} \big(-g\phi_1 + \phi_2\big)g_n \mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t. \tag{21}$$

Furthermore, due to the pointwise convergence and the uniform (in n and t) $L^2(\partial_x M)$–bounds we obtain $g_n \to g$ in $L^2(\mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t)$. By the uniform (and bounded) convergence of F_n to F we can also pass to the limit in the left hand side. Thus,

$$-\int_0^T\int_{I\!R} F(M)\nabla\cdot\boldsymbol{\phi}\mathrm{d}x\mathrm{d}t = \int_0^T\int_{I\!R} \big(-g^2\phi_1 + g\phi_2\big)\mathrm{d}\partial_x M(t,\cdot)\mathrm{d}t, \tag{22}$$

which, after using Lemma 2.5 and (17) once more, leads to

$$-\int_0^T\int_{I\!R} F(M)\nabla\cdot\boldsymbol{\phi}\mathrm{d}x\mathrm{d}t = \int_0^T\int_{I\!R} g\phi_1 \mathrm{d}\partial_t M + \int_0^T\int_{I\!R} g\phi_2 \mathrm{d}\partial_x M. \tag{23}$$

Since $g \in L^2(\partial_x M) \subset L^1(\partial_x M)$ and $g \in L^1(|\partial_t M|)$, we obtain the result. QED.

We are going to need a slightly different result which we now state as a consequence. It may be proved by retracing the proof of Theorem 2.6.

Corollary 2.7. *Assume now that*

$$\partial_t M + \partial_x[\tilde{F}(t,M)] = 0 \text{ in } \mathcal{D}'([0,T]\times I\!R)\ , \tag{24}$$

for some time-linear perturbation $\tilde{F}(t,\cdot) := F + t\Psi$, $\Psi \in C^1[0,1]$. *Then* $F \circ M \in BV([0,T]\times I\!R)$ *and*

$$\nabla[\tilde{F}(t,M)] = (g + t\psi)\nabla M + \mathbf{V}\ , \tag{25}$$

for g *from Theorem 2.6, and the vector field* $\mathbf{V} := \big(\Psi\circ M, 0\big)$. *Here,*

$$\psi(t,x) := \int_0^1 \Psi'\big((1-s)M(t,x-) + sM(t,x)\big)\mathrm{d}s.$$

3 Pressureless Euler/Euler-Poisson systems via scalar conservation laws

3.1 Convergence of measures and their distribution functions

Let $1 \leq p < \infty$, $\mu \in \mathcal{P}_p(I\!R)$ and $v \in L^p(\mu) \cap C(I\!R)$. Then by the de la Vallée-Poussin lemma which can be found in [10], there exists a nonnegative, convex, increasing function $\zeta \in C^1([0,+\infty))$ satisfying $\zeta(0) = 0$ and $\frac{\zeta(t)}{t} \uparrow +\infty$ as $t \to +\infty$ such that

$$\int_{I\!R} \zeta(|x|^p + |v(x)|^p) \mathrm{d}\mu(x) < +\infty.$$

It is also well known that (see [11] for example) there exist a probability space (Ω, Σ, P) and a sequence of independent random variables $\xi_i : \Omega \to I\!R$ such that

$$\xi_{i\#}P = \mu.$$

Now for each positive integer n and each $\omega \in \Omega$, define

$$\mu^{n,\omega} := \frac{1}{n} \sum_{i=1}^{n} \delta_{\xi_i(\omega)}.$$

Then by the Strong Law of Large Numbers and the separability of $C_c(I\!R)$ we have for P-almost every $\omega \in \Omega$,

$$\int_{I\!R} f(x) \mathrm{d}\mu^{n,\omega}(x) = \frac{1}{n} \sum_{i=1}^{n} f(\xi_i(\omega)) \to \mathbb{E}(f \circ \xi_1) = \int_{I\!R} f(x) \mathrm{d}\mu(x) \quad \text{for all functions} \quad f \in C_c(I\!R).$$

Consequently, $\mu^{n,\omega}$ converges narrowly to μ. Thus, by using the Strong Law of Large Numbers again, we also have that for P-almost every $\omega \in \Omega$

$$\int_{I\!R} \zeta(|x|^p) \mathrm{d}\mu^{n,\omega}(x) \to \int_{I\!R} \zeta(|x|^p) \mathrm{d}\mu(x), \quad \int_{I\!R} \zeta(|v|^p) \mathrm{d}\mu^{n,\omega}(x) \to \int_{I\!R} \zeta(|v|^p) \mathrm{d}\mu(x).$$

The last fact together with the properties of ζ yields for $k > 0$ large enough,

$$\frac{\zeta(k^p)}{k^p} \int_{\{x:|x| \geq k\}} |x|^p \mathrm{d}\mu^{n,\omega} \leq \int_{\{x:|x| \geq k\}} \frac{\zeta(|x|^p)}{|x|^p} |x|^p \mathrm{d}\mu^{n,\omega} \leq \int_{I\!R} \zeta(|x|^p) \mathrm{d}\mu^{n,\omega} \leq C(\omega).$$

Therefore, $\{\mu^{n,\omega}\}$ has uniformly integrable p-moments. Consequently, by Proposition 7.1.5 in [2] we obtain that for P-almost every $\omega \in \Omega$, $\mu^{n,\omega} \to \mu$ in W_p and

$$\int_{I\!R} \zeta(|x|^p) \mathrm{d}\mu^{n,\omega}(x) \to \int_{I\!R} \zeta(|x|^p) \mathrm{d}\mu(x), \quad \int_{I\!R} \zeta(|v|^p) \mathrm{d}\mu^{n,\omega}(x) \to \int_{I\!R} \zeta(|v|^p) \mathrm{d}\mu(x).$$

Lemma 3.1. *Suppose $1 \leq p < \infty$ and $\{\mu_n\}$ is a sequence of measures in $\mathcal{P}_p(I\!R)$ converging to $\mu \in \mathcal{P}_p(I\!R)$ in W_p. When $p > 1$ we assume further that $\int_{I\!R} \zeta(|x|^p) \mathrm{d}\mu_n(x)$ is uniformly bounded in n for some nonnegative, convex, increasing function $\zeta \in C^1([0,+\infty))$ satisfying $\zeta(0) = 0$ and $\frac{\zeta(t)}{t} \uparrow +\infty$ as $t \to +\infty$. Then we have* *(i) The L^1 norm of $M_n - M$ on $\mathbb{R}$ goes to zero, where*

$$M(x) := \mu((-\infty, x]).$$

(ii) For any nondecreasing C^1 function B on $[0,1]$ such that $B(0) = 0, B(1) = 1$,

$$\partial_x(B(M_n)) \to \partial_x(B(M)) \quad \text{in} \quad W_p.$$

Proof: From Theorem 6.0.2 in [2], we obtain

$$W_1(\mu_n, \mu) = \int_0^1 |M_n^{-1}(s) - M^{-1}(s)| \mathrm{d}s,$$

where the generalized inverse M^{-1} of M is defined by

$$M^{-1}(s) := \inf\{x \in \mathbb{R} : M(x) > s\}, \quad s \in [0,1].$$

Then by using Fubini's theorem we get $W_1(\mu_n, \mu) = \|M_n - M\|_{L^1(\mathbb{R})}$. This together with the fact $W_1(\mu_n, \mu) \leq W_p(\mu_n, \mu)$ gives (i). By (i) and the Lipschitz condition on B we clearly have $\|B(M_n) - B(M)\|_{L^1(\mathbb{R})} \to 0$. But as

$$W_1(\partial_x(B(M_n)), \partial_x(B(M))) = \|B(M_n) - B(M)\|_{L^1(\mathbb{R})},$$

we infer in particular that $\partial_x(B(M_n)) \to \partial_x(B(M))$ narrowly.

We have from Theorem 2.2

$$\int_{\mathbb{R}} \zeta(|x|^p) \mathrm{d}\partial_x(B(M_n)) = \int_{\mathbb{R}} \zeta(|x|^p) g_n(x) \mathrm{d}\mu_n(x),$$

where $g_n(x) = \int_0^1 B'(sM_n(x) + (1-s)M_n(x-))\mathrm{d}s$. Therefore,

$$\int_{\mathbb{R}} \zeta(|x|^p) \mathrm{d}\partial_x(B(M_n)) \leq \|B'\|_\infty \int_{\mathbb{R}} \zeta(|x|^p) \mathrm{d}\mu_n(x) \leq C\|B'\|_\infty.$$

It follows from this and the properties of the function ζ that for any $k > 0$ large enough,

$$\int_{\{x:|x|\geq k\}} |x|^p \mathrm{d}\partial_x(B(M_n)) \leq \frac{k^p}{\zeta(k^p)} C\|B'\|_\infty.$$

That is, the sequence of probability measures $\partial_x(B(M_n))$ has uniformly integrable p-moments. Consequently, we can conclude from Proposition 7.1.5 in [2] that

$$\partial_x(B(M_n)) \to \partial_x(B(M))$$

in W_p as desired. QED.

3.2 Convergence of the discrete problem

Following [7], we take a discrete probability measure

$$\rho_0^n := \sum_{j=1}^{n} m_j \delta_{x_j},\ x_1 < x_2 < ... < x_n$$

and define

$$\rho^n(t,x) := \sum_{j=1}^{n} m_j \delta_{x_j(t)}\ , \tag{26}$$

where the characteristics are given by

$$x_j(t) = x_j + tv_j + \frac{t^2}{2} a_n\big(M_n(t, x_j(t)-)\big)\ . \tag{27}$$

Here, as in Introduction, we have

$$a_n\big(M_n(t, x_j(t)-)\big) = \left[\alpha\left(\sum_{i=1}^{j} m_i - \frac{1}{2} m_j\right) + \beta\right]\ .$$

We impose the adhesion dynamics at collision (see Introduction) and consider

$$M_n(t,x) := \sum_{j=1}^{n} m_j H\big(x - x_j(t)\big)\ , \tag{28}$$

where H is the right-continuous Heaviside function. Since M_n is piecewise constant, we need only show that M_n solves (4) by checking the Rankine-Hugoniot jump conditions across the shocks $x = x_j(t)$, i.e.

$$\dot{x}_j(t) = \frac{\tilde{F}_n(t, M_n(t, x_j(t))) - \tilde{F}_n(t, M_n(t, x_j(t)-))}{M_n(t, x_j(t)) - M_n(t, x_j(t)-)},\ j = 1, ..., n\ . \tag{29}$$

Assume the masses m_i for $j_0 \le i \le j_1$, including m_j, are all amassed at time t. Then exactly as in [7], we have

$$v_j(t) = \frac{F_n(M_n(t, x_j(t))) - F_n(M_n(t, x_j(t)-))}{M_n(t, x_j(t)) - M_n(t, x_j(t)-)},\ j = 1, ..., n\ ,$$

where F_n is defined by

$$F_n(m) := \int_0^m f_n(\omega) \mathrm{d}\omega \quad \text{for } m \in [0,1].$$

Observe that $\tilde{F}_n(t,m) = F_n(m) + t\int_0^m a_n(\omega)\mathrm{d}\omega$ by the definition of $\tilde{F}_n$ in Introduction. The integrand below is constantly a_n^j on each interval of the form $[M_n(t, x_j(t)-), M_n(t, x_j(t)))$, thus,

$$a_n\big(M_n(t, x_j(t)-)\big) \sum_{i=j_0}^{j_1} m_i = \int_{M_n(t,x_j(t)-)}^{M_n(t,x_j(t))} a_n(\omega)\mathrm{d}\omega$$

implies (29). To prove that M_n is an entropy solution for (4), we check the entropy inequality

$$\dot{x}_j(t) \le \frac{\tilde{F}_n(t,X) - \tilde{F}_n(t, M_n(t, x_j(t)-))}{X - M_n(t, x_j(t)-)} \ , \tag{30}$$

where $X = \sum_{i \le k} m_i$, for some $j_0 \le k \le j_1$. The inequality

$$v_j(t) \le \frac{F_n(X) - F_n(M_n(t, x_j(t)-))}{X - M_n(t, x_j(t)-)}$$

is justified in [7] as a consequence of the *barycentric lemma* (which simply formulates the fact that if two groups of particles collide, then the averaged velocity of the group to the left decreases). Then we see that

$$a_n\big(M_n(t, x_j(t)-)\big) \sum_{i=j_0}^{k} m_i = \int_{M_n(t,x_j(t)-)}^{X} a_n(\omega) \mathrm{d}\omega,$$

which, together with the previous inequality yields (30). We have just sketched the proof of:

Proposition 3.2. *The function M_n given by (28) is the entropy solution of the problem (4).*

Next we want to show that M_n converges in some sense to an entropy solution of (5). The following proposition applies if the initial approximating measures are of the form

$$\rho_0^n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^{(n)}}.$$

Note that Section 3.1 shows that we may consider such approximations.

Proposition 3.3. *Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and consider a sequence of discrete probabilities ρ_0^n as above such that $W_2\big(\rho_0, \rho_0^n\big) \to 0$ and*

$$\int_{\mathbb{R}} \zeta(v_0^2) \mathrm{d}\rho_0^n \to \int_{\mathbb{R}} \zeta(v_0^2) \mathrm{d}\rho_0 < +\infty, \ \int_{\mathbb{R}} \zeta(x^2) \mathrm{d}\rho_0^n \to \int_{\mathbb{R}} \zeta(x^2) \mathrm{d}\rho_0 < +\infty \tag{31}$$

for some nonnegative, convex, increasing function $\zeta \in C^1([0,+\infty))$ satisfying $\zeta(0) = 0$ and $\frac{\zeta(t)}{t} \uparrow +\infty$ as $t \to +\infty$. Let M_0^n and M_0 be the right-continuous cumulative distribution functions of ρ_0^n and ρ_0, respectively. Consider, as above, the entropy solutions M_n of (4) for all n. Then there exists a Borel function $M : [0,\infty) \times \mathbb{R} \to [0,1]$ such that for any given $T > 0$,

$$\max_{0 \le t \le T} W_2\left(\partial_x M_n(t,\cdot), \partial_x M(t,\cdot)\right) \to 0 \ \text{and} \ \max_{0 \le t \le T} \|M_n(t,\cdot) - M(t,\cdot)\|_{L^1(\mathbb{R})} \to 0. \tag{32}$$

Moreover, M is the entropy solution of the problem (5).

Proof: First recall that

$$\rho^n(t,x) := \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j^{(n)}(t)},$$

where

$$x_j^{(n)}(t) = x_j^{(n)} + tv_j^{(n)} + \frac{t^2}{2}\left(\alpha\frac{2j-1}{2n} + \beta\right) .$$

Let $\zeta_t(x) := \zeta(\frac{1}{\kappa}x)$, where κ is some positive constant depending only on t which will be determined later. Then, by the properties of ζ, we have

$$\begin{aligned}
\int_{I\!R} \zeta_t(|x|^2)\mathrm{d}\rho_n(t,x) &= \frac{1}{n}\sum_{j=1}^{n} \zeta\left(\frac{1}{\kappa}|x_j^{(n)}(t)|^2\right) \\
&\le \frac{1}{n}\sum_{j=1}^{n} \zeta\left(\frac{3}{\kappa}\left[|x_j^{(n)}|^2 + t^2|v_j^{(n)}|^2 + \frac{(|\alpha|+|\beta|)^2 t^4}{4}\right]\right) \\
&\le \frac{1}{3}\zeta\left(\frac{9(|\alpha|+|\beta|)^2 t^4}{4\kappa}\right) + \frac{1}{3n}\sum_{j=1}^{n}\zeta\left(\frac{9}{\kappa}|x_j^{(n)}|^2\right) + \frac{1}{3n}\sum_{j=1}^{n}\zeta\left(\frac{9t^2}{\kappa}|v_j^{(n)}|^2\right).
\end{aligned}$$

Hence, by choosing $\kappa = \max\{9, 9t^2\}$ and using again the fact that ζ is increasing, we obtain

$$\int_{I\!R} \zeta_t(|x|^2)\mathrm{d}\rho_n(t,x) \le C_t + \frac{1}{3}\int_{I\!R}\zeta(|x|^2)\mathrm{d}\rho_0^n(t,x) + \frac{1}{3}\int_{I\!R}\zeta(|v_0|^2)\mathrm{d}\rho_0^n(t,x) \le C_t$$

uniformly in n (the last inequality is due to (31)). Since ζ_t has superlinear growth (it is just an argument- rescaled version of ζ), there exists $\rho(t,\cdot) \in \mathcal{P}_2(I\!R)$ such that, up to a subsequence that may depend on t, $W_2(\rho^n(t,\cdot), \rho(t,\cdot)) \to 0$ as $n \to \infty$. By a standard diagonal argument we can choose a subsequence independent of t satisfying $W_2(\rho^n(t,\cdot), \rho(t,\cdot)) \to 0$ for all $t \in [0,\infty) \cap \mathbb{Q}$. In order to see that this conclusion also holds for all t in $[0,\infty)$, we are going to show that the paths $\rho^n(t,\cdot)$ are uniformly local Lipschitz in t. Indeed, let $T > 0$ and $t, s \in [0,T]$ be arbitrary. Then since

$$|x_j^{(n)}(t) - x_j^{(n)}(s)|^2 \le C|t-s|^2\left(|v_j^{(n)}|^2 + T^2\right)$$

we have

$$\begin{aligned}
W_2^2(\rho^n(t,\cdot),\rho^n(s,\cdot)) &\le \frac{1}{n}\sum_{j=1}^{n}|x_j^{(n)}(t) - x_j^{(n)}(s)|^2 \le C|t-s|^2\left(\frac{1}{n}\sum_{j=1}^{n}|v_j^{(n)}|^2 + T^2\right) \\
&= C|t-s|^2\left(\int_{I\!R}|v_0(x)|^2\mathrm{d}\rho_0^n + T^2\right) \le C|t-s|^2 \quad \text{uniformly in } n.
\end{aligned}$$

Using this uniformly Lipschitz property, we can conclude that, in fact,

$$W_2(\rho^n(t,\cdot),\rho(t,\cdot)) \to 0 \text{ as } n \to \infty, \text{ uniformly in } t \in [0,T] \; , \tag{33}$$

which yields (32) with $M(t,x) := \rho(t,(-\infty,x])$.
Next we show that M is a solution of (5). By the assumptions and Lemma 3.1, we have

$$\int_{I\!R} |M_0^n - M_0|\mathrm{d}x \to 0,$$

and

$$\partial_x(B(M_0^n)) \to \partial_x(B(M_0)) \quad \text{in} \quad W_2$$

for any nondecreasing C^1 function B on $[0,1]$ sasisfying $B(0)=0$ and $B(1)=1$. But as v_0 satisfies the assumption $(H2)$, we obtain

$$\int_{I\!R} v_0(x)\mathrm{d}\partial_x(B(M_0^n)) \to \int_{I\!R} v_0(x)\mathrm{d}\partial_x(B(M_0)).$$

By the definitions of f_n and f, this means that

$$\int_0^1 f_n(m)B'(m)\mathrm{d}m \to \int_0^1 f(m)B'(m)\mathrm{d}m$$

for all nondecreasing C^1 functions B on $[0,1]$ satisfying $B(0)=0$ and $B(1)=1$. Indeed,

$$\begin{aligned}\int_{I\!R} v_0(x)\mathrm{d}\partial_x(B(M_0^n))(x) &= \sum_{i=1}^n v_0(x_i^{(n)})\big[B(M_0^n(x_i^{(n)})) - B(M_0^n(x_i^{(n)}-))\big] \\ &= \sum_{i=1}^n \int_{M_0^n(x_i^{(n)}-)}^{M_0^n(x_i^{(n)})} v_0\circ N_0^n(m)B'(m)\mathrm{d}m = \int_0^1 f_n(m)B'(m)\mathrm{d}m.\end{aligned}$$

For the continuous version we use (7) to conclude in a similar way. It then follows that

$$\int_0^1 f_n(m)g(m)\mathrm{d}m \to \int_0^1 f(m)g(m)\mathrm{d}m \quad \text{for all functions} \quad g\in C([0,1]).$$

This together with the fact

$$\int_0^1 |f_n(m)|\mathrm{d}m = \int_{I\!R} |v_0(x)|\mathrm{d}\rho_0^n \le C \quad \text{uniformly in } n$$

yields $f_n \to f$ weakly in L^1. By using the uniform convergence of $\tilde{F}_n$ to $\tilde{F}$ as continuous and bounded functions, we deduce M is a solution of the problem (5). Now for a fixed $t\ge 0$, let U be any C^1 function on $[0,1]$. Define

$$\begin{aligned}\tilde{F}_{n,U}(t,m) &:= \int_0^m [f_n(\omega)+ta_n(\omega)]\,U'(\omega)\mathrm{d}\omega, \\ \tilde{F}_U(t,m) &:= \int_0^m [f(\omega)+ta(\omega)]\,U'(\omega)\mathrm{d}\omega.\end{aligned}$$

Now we use the uniform convergence of a_n to $\alpha\mathrm{id}+\beta$, the facts $f_n\to f$ weakly in L^1 and

$$\int_0^1 \zeta(|f_n|)\mathrm{d}m = \int_{I\!R} \zeta(|v_0|)\mathrm{d}\rho_0^n \le C \tag{34}$$

to deduce that $\tilde{F}_{n,U}$ converges to $\tilde{F}_U$ uniformly in $(t,m)\in[0,T]\times[0,1]$. (Notice that we have used the assumption $(H2)$ to derive (34), the equi-integrability of the sequence $\{f_n\}$ in $L^1([0,1])$.) Therefore, as in [7], we conclude that M is, in fact, an entropy solution for (5). QED.

3.3 The existence result

We finally have all necessary tools to prove Theorem 1.1. Note, however, that we leave out the proof of the fact that the initial conditions are satisfied. We will show that in Proposition 4.9.
Proof: (of Theorem 1.1) Since M is a solution of the problem (5) by Proposition 3.3, we can use Corollary 2.7 to conclude that $\partial_t M + v\partial_x M = 0$, where $v(t,\cdot) := g(t,\cdot) + t\psi(t,\cdot)$ is well-defined $\partial_x M(t,\cdot) =: \rho(t,\cdot)$–a.e. By differentiation in the sense of distributions we obtain the first equation in (1). Then, Corollary 2.7 also gives

$$\begin{aligned}\partial_t(\rho v) &= \partial_t\big[\partial_x[\tilde F(t,M)]\big] = \partial_x\big[\partial_t[\tilde F(t,M)]\big]\\ &= \partial_x[v\partial_t M + \Psi(M)] = \partial_x(-v^2\rho) + \psi\rho.\end{aligned}$$

Note that, in our case, $\Psi(m) = \alpha m^2/2 + \beta m$. Thus,

$$\psi(t,x) = \beta + \alpha\int_0^1 [(1-s)M(t,x-) + sM(t,x)]\mathrm{d}s = \beta + \alpha[M(t,x) - \frac{1}{2}\rho(t,\{x\})].$$

Since $\partial_x M(t,\cdot) = \rho(t,\cdot)$ and $\rho(t,\cdot)$ has at most countably many atoms, we may take $\Phi(t,x) = \int_{-\infty}^x M(t,y)\mathrm{d}y$ to conclude. QED.

Remark 3.4. *Note that the term $M(t,x) - \frac{1}{2}\rho(t,\{x\})$ is precisely the barycentric projection [2] onto $\rho(t,\cdot)$ of the optimal coupling between $\chi_{(0,1)}$ and $\rho(t,\cdot)$. It differs from the projection considered in [14] by an additive factor of 0.5 due to the fact that, instead of $\chi_{(0,1)}$, the reference measure in [14] was $\chi_{(-0.5,0.5)}$.*

4 Time regularity, entropy condition, and shocks

In this section we discuss some qualitative properties of the "sticky particle" solution. The family of absolutely continuous curves in $\mathcal{P}_2(I\!R)$ is central to our approach. Thus, recall that $(\mathcal{P}_2(I\!R), W_2)$ is a Polish space on which we define absolutely continuous curves by saying that $[0,T] \ni t \to \mu_t \in \mathcal{P}_2(I\!R)$ lies in $AC^2(0,T;\mathcal{P}_2(I\!R))$ provided that there exists $f \in L^2(0,T)$ such that $W_2(\mu_t, \mu_{t+h}) \le \int_t^{t+h} f(s)ds$ for all $0 < t < t+h < T$.

4.1 The solution path is locally Lipschitz

First we show that our solution path is locally Lipschitz.

Proposition 4.1. *The solution path $t \to \rho(t,\cdot)$ satisfies*

(i) For any $0 < T < +\infty$, we have

$$W_2(\rho(t,\cdot),\rho(s,\cdot)) \le C_T|t-s| \quad \text{for all } t,s \in [0,T].$$

(ii) The energy is nonincreasing, i.e.

$$\int_{I\!R} |v(t,x)|^2 \mathrm{d}\rho(t,x) \le \int_{I\!R} |v_0(x)|^2 \mathrm{d}\rho_0(x) \quad \text{for all } t \ge 0.$$

Proof: Recall that $\rho^n(t,\cdot)$ are uniformly Lipschitz in n and t. Thus, by the triangle inequality,

$$W_2(\rho(t,\cdot),\rho(s,\cdot)) \leq W_2(\rho(t,\cdot),\rho^n(t,\cdot)) + C|t-s| + W_2(\rho^n(s,\cdot),\rho(s,\cdot)).$$

Therefore, by letting n go to infinity and using Proposition 3.3 we obtain (i). Also as $v_n(t,\cdot)\rho^n(t,\cdot) \to v(t,\cdot)\rho(t,\cdot)$ weakly for each t, we deduce that

$$\begin{aligned}\int_{I\!R} |v(t,x)|^2 \mathrm{d}\rho(t,x) &\leq \liminf_{n\to\infty} \int_{I\!R} |v_n(t,x)|^2 \mathrm{d}\rho^n(t,x) \leq \liminf_{n\to\infty} \int_{I\!R} |v_0(x)|^2 \mathrm{d}\rho_0^n(x) \\ &= \int_{I\!R} |v_0(x)|^2 \mathrm{d}\rho_0(x).\end{aligned}$$

QED.

Remark 4.2. *In particular, $\rho \in AC^2(0,T;\mathcal{P}_2(I\!R))$.*

We now recall a result, slightly modified, proved in [14].

Proposition 4.3. *Suppose $\sigma \in AC^2(0,T;\mathcal{P}_2(I\!R))$. Let v be the velocity of minimal norm associated to σ and $N(t,\cdot) : (0,1) \to I\!R$ be the monotone nondecreasing map such that $N(t,\cdot)_{\#}\chi_{(0,1)} = \sigma(t,\cdot)$. For each t, modifying $N(t,\cdot)$ on a countable subset of $(0,1)$ if necessary, we may assume without loss of generality that $N(t,\cdot)$ is right-continuous. Then, $N \in H^1(0,T;L^2(0,1))$ and*

$$\dot{N}(t,x) = v(t,N(t,x)) \tag{35}$$

for $\mathcal{L}^2$-almost every $(t,x) \in (0,T)\times(0,1)$.

Note that the *minimal norm* assumption is, in fact, redundant. Indeed, we prove

Lemma 4.4. *Consider the path $t \to \mu \in AC^2(0,T;\mathcal{P}_2(I\!R))$ for some $0 < T < +\infty$. Then the velocity defined in [2] (called "of minimal norm") is the unique velocity along the curve μ in the following sense: if*

$$\partial_t\mu + \partial_x(\mu v_i) = 0 \text{ in } \mathcal{D}'([0,T]\times I\!R),\ i = 1,2$$

for some v_i Borel measurable in (t,x) such that $v_i(t,\cdot) \in L^2(\mu(t,\cdot))$ for Lebesgue a.e. $t \in (0,T)$, then for Lebesgue a.e. $t \in (0,T)$ we have $v_1(t,\cdot) \equiv v_2(t,\cdot)$ in the $\mu(t,\cdot)$–a.e. sense.

Proof: By subtraction and by taking test functions $\varphi(t,x) = \xi(t)\zeta(x)$, the equations above readily yield

$$\int_{I\!R} u(t,x)\zeta'(x)\mathrm{d}\mu(t,x) = 0 \text{ for a.e. } t \in (0,T) \text{ and any } \zeta \in C_c^1(I\!R),$$

where $u := v_1 - v_2$. Fix $\varepsilon > 0$ and $\phi \in C_c(I\!R)$. If $\phi = 0$ on $[R,+\infty)$, consider, for each natural number $n > R$, the function

$$\Phi_n(x) := \begin{cases} \int_{-\infty}^x \phi(y)\mathrm{d}y & \text{if } x < n, \\ \omega(x-n) & \text{if } n \leq x \leq n+1, \\ 0 & \text{if } x > n+1 \end{cases} \tag{36}$$

where $\omega \in C^1[0,1]$ such that $\omega(0) = \int_{-\infty}^R \phi(y)\mathrm{d}y$, $\omega(1) = 0$ and $\omega'(0) = 0 = \omega'(1)$. Clearly, $\Phi_n \in C_c^1(I\!R)$. Thus,

$$\int_{I\!R} u(t,x)\phi(x)\mathrm{d}\mu(t,x) + \int_n^{n+1} u(x,t)\omega'(x-n)\mathrm{d}\mu(t,x) = 0 \text{ for a.e. } t \in (0,T) \ .$$

We have $|\omega'(x-n)| \le \|\omega'\|_{L^\infty(0,1)} =: C$ for all $n > R$ and all $x \in (n, n+1)$. Since $\mu(t,\cdot)$ is a Borel probability for Lebesgue a.e. $t \in (0,T)$, we conclude that for such t we have

$$\left| \int_{I\!R} u(t,x)\phi(x)\mathrm{d}\mu(t,x) \right| \le C\|u(t,\cdot)\|_{L^2(\mu(t,\cdot))}\mu(t,[n,n+1])^{1/2} \le \varepsilon$$

if n is sufficiently large. Due to the arbitrariness of ε and ϕ, the proof is concluded. QED.

Remark 4.5. *Note that, in fact, we have just proved that* $\{\varphi' : \varphi \in C_c^\infty(I\!R)\}$ *is dense in* $L^2(\mu)$*, even though* μ *may not necessarily have finite p-order moment (for any* $p > 0$*). Also, as a consequence, the tangent space* $\mathcal{T}_\mu\mathcal{P}_2(I\!R)$ *[2] is the whole* $L^2(\mu)$*. This property was brought to our attention by W. Gangbo.*

4.2 Recovery of the entropy condition

We shall now prove that the solution we obtained for (1) in the $\alpha = 0 = \beta$ case satisfies the *Oleinik entropy condition*, i.e.

Theorem 4.6. *For Lebesgue almost all* $t \in (0,T)$ *we have*

$$v(t,x_2) - v(t,x_1) \le \frac{1}{t}(x_2 - x_1) \textit{ for } \rho(t,\cdot) - a.e.\ x_1 \le x_2. \tag{37}$$

Proof: Let us look back at the discrete problem. Note that

$$N_n(t,\omega) := x_j(t) \text{ whenever } M_n(t,x_j(t)-) \le \omega < M_n(t,x_j(t))$$

is the optimal map such that $N_n(t,\cdot)_\#\chi_{(0,1)} = \rho^n(t,\cdot)$. It is known [7], [16] that the discrete problem satisfies the Oleinik entropy condition, i.e.

$$t\big[v_n(t,x_{i_2}(t)) - v_n(t,x_{i_1}(t))\big] \le x_{i_2}(t) - x_{i_1}(t) \text{ whenever } i_1 \le i_2.$$

Since $v_n(t,x_j(t)) = \dot{x}_j(t)$ away from collision times, we infer that the map $t \to \big[x_{i_2}(t) - x_{i_1}(t)\big]/t$ is piecewise nonincreasing. But this map is continuous, so it is globally nonincreasing, and, due to the definition of N_n, it follows that

$$t \to \frac{1}{t}\big[N_n(t,\omega_2) - N_n(t,\omega_1)\big] \tag{38}$$

is nonincreasing in $(0,T)$ for all $\omega_1 \le \omega_2 \in (0,1)$. Now let $\Delta := \{\omega = (\omega_1,\omega_2) \in [0,1]^2 : \omega_1 \le \omega_2\}$ and $S_n(t,\omega) = N_n(t,\omega_2) - N_n(t,\omega_1)$ be defined on $[0,T]\times\Delta$. It is easy to see that $S_n \in H^1(0,T;L^2(\Delta))$ and that (38) implies

$$\int_0^T\int_\Delta \frac{S_n(t,\omega)}{t}\partial_t\varphi(t,\omega)\mathrm{d}\omega\mathrm{d}t \ge 0, \text{ for all nonnegative } \varphi \in C_c^\infty((0,T)\times\Delta). \tag{39}$$

On the other hand, due to (32), we infer that

$$\|N_n - N\|_{L^1((0,T)\times(0,1))} = \int_0^T W_1(\rho^n(t,\cdot),\rho(t,\cdot))\mathrm{d}t \to 0,$$

where $N(t,\cdot)$ is the optimal map such that $N(t,\cdot)_{\#}\chi_{(0,1)} = \rho(t,\cdot)$. In particular, $(S_n/t) \to (S/t)$ in $\mathcal{D}'((0,T)\times\Delta)$, where $S(t,\omega) = N(t,\omega_2) - N(t,\omega_1)$ if $\omega \in \Delta$. Thus, (39) implies

$$\int_0^T \int_\Delta \frac{S(t,\omega)}{t}\partial_t\varphi(t,\omega)\mathrm{d}\omega\mathrm{d}t \geq 0, \text{ for all nonnegative } \varphi \in C_c^\infty((0,T)\times\Delta).$$

Therefore, $\partial_t[S/t] \leq 0$ in the distributional sense, which implies $t\dot{S} - S \leq 0$ in the $\mathcal{L}^3$–a.e. sense, i.e.

$$\dot{N}(t,\omega_2) - \dot{N}(t,\omega_1) \leq \frac{1}{t}\big[N(t,\omega_2) - N(t,\omega_1)\big] \text{ for Lebesgue a.e. } (t,\omega) \in (0,T)\times\Delta.$$

Consequently, (35) yields

$$v(t,N(t,\omega_2)) - v(t,N(t,\omega_1)) \leq \frac{1}{t}\big[N(t,\omega_2) - N(t,\omega_1)\big] \text{ for Lebesgue a.e. } (t,\omega) \in (0,T)\times\Delta,$$

which, due to $N(t,\cdot)_{\#}\chi_{(0,1)} = \rho(t,\cdot)$, implies (37). QED.

4.3 Formation of shocks

Now let us assume ρ_0 is atom-free and take N_0 be the optimal map pushing $\chi_{(0,1)}$ forward to ρ_0.

Proposition 4.7. *Let $T := \sup\{t \in [0,\infty) : \mathrm{id} + tv_0$ is nondecreasing on $\mathrm{spt}(\rho_0)\}$. If $T = 0$, then the solution develops atomic singularities instantaneously. If $0 < T < +\infty$, then the solution remains non-atomic before T and develops atomic singularities instantaneously after T. If $T = +\infty$, then the solution is atom-free at all times.*

Proof: Let us treat the case $0 < T < +\infty$. Indeed, it will become clear that the other two cases can be handled almost identically. Note that, due to the definition of T and the fact that ρ_0 is non-atomic, $\mathrm{id} + tv_0$ is (strictly) increasing on $\mathrm{spt}(\rho_0)$ for $t \in [0,T)$. Thus, $N_0 + tv_0 \circ N_0$ is increasing on $(0,1)$ for $t \in [0,T)$. It follows that

$$\bar{M}(t,\cdot) := (N_0 + tv_0 \circ N_0)^{-1} = M_0 \circ (\mathrm{id} + tv_0)^{-1}$$

is the entropy solution (given by characteristics) of

$$\partial_t\bar{M} + \partial_x[F(\bar{M})] = 0, \quad \bar{M}(0,\cdot) = M_0\ ,$$

where $F' = v_0 \circ N_0$, $F(0) = 0$. Due to uniqueness of the entropy solution, we get $\bar{M} \equiv M$. Thus, $\rho(t,\cdot) = (\mathrm{id}+tv_0)_{\#}\rho_0$ for $t \in [0,T]$ (this is precisely the geodesic connecting ρ_0 and ρ_T in $\mathcal{P}_2(I\!R)$). Again, since $\mathrm{id} + tv_0$ is (strictly) increasing on $\mathrm{spt}(\rho_0)$ for $t \in [0,T)$, we deduce $\rho(t,\cdot)$

has no atoms if $t \in [0,T)$. To conclude, we argue by contradiction and suppose $\rho(t,\cdot)$ is atom-free on $[0,T+\epsilon]$ for some $\epsilon > 0$. Thus, $M(t,\cdot)$ is continuous and so $v(t,\cdot) = v_0 \circ N_0 \circ M(t,\cdot)$ for all $t \in [0,T+\epsilon]$. Since $M(t,\cdot)$ is now the optimal map pushing $\rho(t,\cdot)$ forward to $\chi_{(0,1)}$, we infer

$$v(t,N(t,m)) = v_0 \circ N_0 \circ M(t,N(t,m)) = v_0 \circ N_0(m) \text{ a.e. } m \in (0,1),$$

which, in light of (35), yields $\dot{N}(t,m) = v_0 \circ N_0(m)$ for a.e. $(t,m) \in (0,T+\epsilon) \times (0,1)$. Thus, as an $H^1(0,T+\epsilon; L^2(0,1))$ map, $N(t,\cdot) = N_0 + tv_0 \circ N_0$. In particular, we obtain that $\mathrm{id} + tv_0$ is nondecreasing on the support of ρ_0 for all $t \in [0,T+\epsilon]$, which contradicts the definition of T. Therefore, the solution becomes atomic instantaneously after T. QED.

Remark 4.8. *What happens at* $t = T$ *depends on whether or not* $\mathrm{id} + Tv_0$ *has "flat spots" on* $\mathrm{spt}(\rho_0)$. *It is easy to construct examples illustrating that each of these situations may occur.*

4.4 Continuity of the energy and a remark on uniqueness

Proposition 4.9. *Suppose* (ρ_0, v_0) *satisfies the conditions* $(H1)$ *and* $(H2)$. *Let* (ρ, v) *be the weak solution to the system (1) given in the proof of Theorem 1 in subsection 3.3. Then* (ρ, v) *has the following property:*

$$\lim_{t\to 0^+} \int_{I\!R} v(t,x)\varphi(x)\mathrm{d}\rho(t,x) = \int_{I\!R} v_0(x)\varphi(x)\mathrm{d}\rho_0(x) \quad \textit{for all} \quad \varphi \in C_b(I\!R), \tag{40}$$

which shows that the initial condition for the velocity is satisfied. Moreover, we have

$$\lim_{t\to 0^+} \int_{I\!R} v^2(t,x)\varphi(x)\mathrm{d}\rho(t,x) = \int_{I\!R} v_0^2(x)\varphi(x)\mathrm{d}\rho_0(x) \quad \textit{for all} \quad \varphi \in C_b(I\!R). \tag{41}$$

Proof: Due to $\|M(t,\cdot) - M_0\|_{L^1(I\!R)} = W_1(\rho(t,\cdot),\rho_0) \to 0$ and the BV calculus, we have for any function $\varphi \in C_c^\infty(I\!R)$

$$\begin{aligned} \lim_{t\to 0^+} \int_{I\!R} v(t,x)\varphi(x)\mathrm{d}\rho(t,x) &= -\lim_{t\to 0^+} \int_{I\!R} \varphi'(x)F(M(t,x))\mathrm{d}x \\ = -\int_{I\!R} \varphi'(x)F(M_0(x))\mathrm{d}x &= \int_{I\!R} \varphi(x)g(x)\mathrm{d}\rho_0(x), \end{aligned} \tag{42}$$

where $g(x)$ is given by

$$g(x) = \int_0^1 v_0 \circ N_0\left((1-s)M_0(x-) + sM_0(x)\right)\mathrm{d}s, \quad x \in I\!R.$$

We claim that $N_0\left((1-s)M_0(x-) + sM_0(x)\right) = x$ for ρ_0-a.e. $x \in I\!R$. Indeed, if a point x satisfies $\rho_0(\{x\}) \neq 0$ then M_0 has a jump at x. Therefore, it is clear in this case that

$$N_0\left((1-s)M_0(x-) + sM_0(x)\right) = \inf\left\{z : M_0(z) > (1-s)M_0(x-) + sM_0(x)\right\} = x.$$

As M_0 has at most countably many "flat spots", the claim shall be proved if we can show that $g(x) = v_0(x)$ whenever x satisfies $x \in \mathrm{spt}(\rho_0)$, $\rho_0(\{x\}) = 0$ and x is not one of the endpoints of some flat spot of M_0. To see this is true, let x be a point with such properties. Then there exists $\epsilon > 0$ such that the function M_0 is strictly increasing on $(x-\epsilon, x+\epsilon)$. It follows that

$$N_0\left((1-s)M_0(x-) + sM_0(x)\right) = N_0\left(M_0(x)\right) = x.$$

Thus, we obtain the claim, which in turn yields $g(x) = v_0(x)$ for ρ_0-a.e. $x \in I\!\!R$. Hence, by combining with (42), we get

$$\lim_{t\to 0^+} \int_{I\!\!R} v(t,x)\varphi(x)\mathrm{d}\rho(t,x) = \int_{I\!\!R} v_0(x)\varphi(x)\mathrm{d}\rho_0(x) \quad \text{for all} \quad \varphi \in C_c^\infty(I\!\!R)\ .$$

By a simple approximation and the fact $\|v_0\|_{L^2(\rho_0)}$ is finite, this gives (40). As a consequence of (40) and the fact that the energy is nonincreasing proved in Proposition 4.1, we obtain (see, e.g. Theorem 5.4.4 in [2])

$$\lim_{t\to 0^+} \int_{I\!\!R} v^2(t,x)\mathrm{d}\rho(t,x) = \int_{I\!\!R} v_0^2(x)\mathrm{d}\rho_0(x).$$

In addition, we have $W_2(\rho(t,\cdot),\rho_0) \le Ct$ from Proposition 4.1. Therefore, by using Proposition 7.1.5 and Theorem 5.4.4 in [2] we can conclude that (41) holds. QED.

We end the paper by the following remark on the uniqueness of the solution.

Remark 4.10. *If we assume in addition that v_0 is bounded, then the weak solution (ρ, v) constructed above is the unique weak solution satisfying the entropy conditon in Theorem 4.6 and the weak convergence of $v^2(t,\cdot)\,\rho(t,\cdot)$ to $v_0^2\rho_0$ (such weak solution is called an entropy solution for pressureless gases [17]). This fact follows directly from Theorem 4.6, Proposition 4.9 and the uniqueness result in [17]. Thus, if $v_0 \in C_b(I\!\!R)$, we have proved that any entropy solution to the pressureless gases system can be obtained as a weak limit of sticky particles. We note that a similar result was obtained by Bouchut and James in [5] for duality solutions.*

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