Composing and covering of coalgebras.

Stefan Forcey, U. Akron
Aaron Lauve, Loyola U. Chicago
Frank Sottile, Texas A&M U.

August 30, 2010

“Y ese color azul que sólo es permitido ver en los sueños.” - Borges
The divided power Hopf algebra $\mathcal{C}Sym$ is ordinarily presented as follows:

$$\mathbb{K}[x] := \text{span}\{x^{(n)} : n \geq 0\}$$

with basis vectors $x^{(n)}$ satisfying

$$x^{(m)} \cdot x^{(n)} = \binom{m+n}{n} x^{(m+n)}$$

and

$$\Delta(x^{(n)}) = \sum_{i+j=n} x^{(i)} \otimes x^{(j)}.$$

Example:

$$x^{(2)} \cdot x^{(3)} = 10x^{(5)}$$
A Hopf algebra of binary trees [LR].

Two operations on trees: Splitting

And grafting:
The $n^{th}$ component of $\mathcal{Y}Sym$ has basis the collection of binary trees with $n$ interior nodes, and thus $n + 1$ leaves, denoted $\mathcal{Y}_n$. 
The $n^{th}$ component of $\mathcal{Y}Sym$ has basis the collection of binary trees with $n$ interior nodes, and thus $n + 1$ leaves, denoted $\mathcal{Y}_n$.

Here is the coproduct:

$$\Delta \mathcal{Y} = 1 \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes 1$$
Loday–Ronco Hopf algebra.

The $n^{th}$ component of $\mathcal{Y}Sym$ has basis the collection of binary trees with $n$ interior nodes, and thus $n + 1$ leaves, denoted $\mathcal{Y}_n$.

Here is the coproduct:

\[ \Delta \mathcal{Y} = I \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} \]

Here is how to multiply two trees:

\[ \mathcal{Y} \ast \mathcal{Y} = \mathcal{Y} + \mathcal{Y} \ast \mathcal{Y} + \mathcal{Y} \ast \mathcal{Y} + \mathcal{Y} \ast \mathcal{Y} \]
The $n^{th}$ component of $\mathcal{YSym}$ has basis the collection of binary trees with $n$ interior nodes, and thus $n + 1$ leaves, denoted $\mathcal{Y}_n$.

Here is the coproduct:

\[
\Delta \mathcal{Y} = \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} = \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y}.
\]

Here is how to multiply two trees:

\[
\mathcal{Y} \cdot \mathcal{Y} = \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y} + \mathcal{Y} \otimes \mathcal{Y}.
\]
The long way to multiply in $\mathcal{C}Sym$.

We draw basis elements of $\mathcal{C}Sym$ as *right combs*. $x^{(4)} = \begin{tikzpicture}[x=0.5ex,y=0.5ex] \draw[thick] (0,0) -- (4,0); \draw[thick] (0,0) -- (0,4); \draw[thick] (0,0) -- (2,4); \draw[thick] (4,0) -- (2,4); \end{tikzpicture}$.
The long way to multiply in $\mathcal{C}Sym$.

We draw basis elements of $\mathcal{C}Sym$ as right combs. $x^{(4)} = \bigotimes \bigotimes$. 

Here is the coproduct:

$$\Delta = \bigotimes \bigotimes + \bigotimes \bigotimes + \bigotimes \bigotimes$$
The long way to multiply in $C_{Sym}$.

We draw basis elements of $C_{Sym}$ as *right combs*. $x^{(4)} = \begin{array}{c}
\end{array}$.

Here is the coproduct:

$\Delta = \begin{array}{c}
\end{array}$

Here is how to multiply two combs:

$\begin{array}{c}
\end{array}$
The long way to multiply in $\mathcal{C}Sym$.

We draw basis elements of $\mathcal{C}Sym$ as *right combs*. $x^{(4)} = \vphantom{Y}$. 

Here is the coproduct:

\[
\Delta \quad = \quad Y \otimes Y + Y \otimes Y + Y \otimes Y
\]

Here is how to multiply two combs:

\[
\cdot \quad = \quad Y + Y + Y + Y
\]
The idea is that given two graded coalgebras we can combine them in a way reminiscent of operad composition. Let $\mathcal{C}$ and $\mathcal{D}$ be two graded coalgebras. We will form a new coalgebra $\mathcal{E} = \mathcal{C} \circ \mathcal{D}$ on the vector space

$$\mathcal{C} \circ \mathcal{D} := \bigoplus_{n \geq 0} \mathcal{C}^{\otimes (n+1)} \otimes \mathcal{D}_n.$$ (1)
The motivating example is when $\mathcal{C}$ and $\mathcal{D}$ are spaces of rooted trees. Then $\circ$ may be interpreted as some rule for grafting $n+1$ trees from $\mathcal{C}$ onto the leaves of a tree in $\mathcal{D}_n$.

**Example**

Suppose $\mathcal{C} = \mathcal{D} = \mathcal{Y} \text{Sym}$ and consider some $(c_0, \ldots, c_n) \times d \in (\mathcal{Y}^{n+1}) \times \mathcal{Y}_n$. Then defining $\circ$ by grafting with color coding, e.g.,

$$(\begin{array}{cccc}
\mathcal{Y}
, & \mathcal{I}
, & \mathcal{Y}
, & \mathcal{Y}
, & \mathcal{I}
\end{array}) \times \begin{array}{c}
\mathcal{Y}
\end{array} \xrightarrow{\circ}$$

gives a new flavor of tree: *painted trees*. 
A small commuting diamond

Stefan Forcey, Aaron Lauve, Frank Sottile, Composing and covering of coalgebras.
A small commuting diamond

Stefan Forcey, Aaron Lauve, Frank Sottile, Composing and covering of coalgebras.
Let $\mathcal{D}$ be a graded bialgebra, and thus a Hopf algebra.

**Definition**

A *covering coalgebra* of $\mathcal{D}$ is a graded coalgebra $\mathcal{E}, \Delta_\mathcal{E}$ together with a coalgebra map $f: \mathcal{E} \to \mathcal{D}$ for which we have:

- $\mathcal{E}$ is a graded module over $\mathcal{D}$. We denote the action by $*: \mathcal{E} \otimes \mathcal{D} \to \mathcal{E}$.
- The coalgebra map $f$ is also a module map.
- The action and coproducts agree.
Our compositions of tree-like algebras serve as prime examples!
A covering coalgebra turns out to give us a lot of extra free structure.

**Theorem**

A covering coalgebra has the structures of a one-sided Hopf algebra, a Hopf module and a comodule algebra.

**Proof.**

For the one-sided Hopf algebra, we define the product $m_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}$ by $m_{\mathcal{E}} = \star \circ (1 \otimes f)$. The one-sided unit is $1_{\mathcal{E}}$. For the other two structures we need a coaction $\rho : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{D}$. This coaction is defined by $\rho = (1 \otimes f) \circ \Delta_{\mathcal{E}}$.

\[ \Box \]
Examples from trees.

Here is an example of the coproduct in $\mathcal{V}Sym \circ \mathcal{V}Sym$:

\[
\Delta = Y \otimes Y + Y \otimes Y + Y \otimes Y + Y \otimes Y
\]

Here is an example of the product in $\mathcal{V}Sym \circ \mathcal{V}Sym$:

\[
\bullet = Y + Y + Y + Y
\]
$\mathcal{C} Sym \circ \mathcal{C} Sym$ is indexed by vertices of the cubes, represented by trees formed by grafting a forest of combs to the leaves of a comb (we paint the edges of the latter.)

There is a simple way to associate one of these trees with a composition.

We write the composition as the string $k_0 k_1 \ldots k_j$. The number $k_i$ is just the number of leaves of the unpainted comb grafted to the $i^{th}$ painted edge.
The coproduct is the usual splitting of trees:

\[ \Delta \ F_{13} = \]

\[ \Delta = F_1 \otimes F_{13} + F_{11} \otimes F_3 + F_{12} \otimes F_2 + F_{13} \otimes F_1 \]

Here is the product:

\[ F_{13} \cdot F_2 = \]

\[ F_{113} + F_{113} + F_{122} + F_{131} \]
Geometry and combinatorics.

There are interesting combinatorial and geometric constructions which run in parallel to our algebraic composition of algebras.
polytopes.
polytopes.
polytopes.
polytopes.
polytopes.
polytopes.
More polytopes.

Stefan Forcey, Aaron Lauve, Frank Sottile,

Composing and covering of coalgebras.
More polytopes.
Future

More work:
• primitives and coinvariants.
• Formulas for antipodes.
Questions and comments?