Composing and covering of coalgebras.

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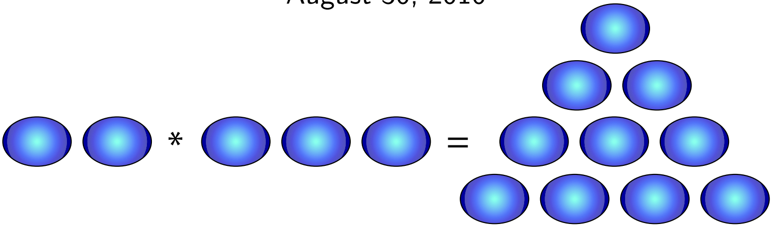


“Y ese color azul que sólo es permitido ver en los sueños.” -*Borges*

The divided power Hopf algebra.

The divided power Hopf algebra $\mathfrak{C}Sym$ is ordinarily presented as follows:

$$\mathbb{K}[x] := \mathrm{span}\{x^{(n)} : n \geq 0\}$$

with basis vectors $x^{(n)}$ satisfying

$$x^{(m)} \cdot x^{(n)} = \binom{m+n}{n} x^{(m+n)}$$

and

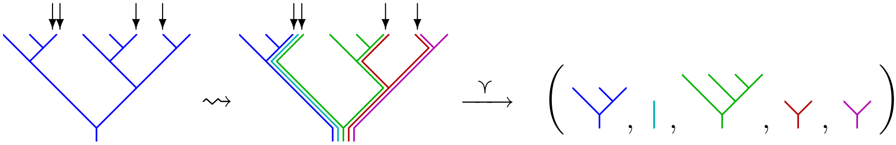
$$\Delta(x^{(n)}) = \sum_{i+j=n} x^{(i)} \otimes x^{(j)}.$$

Example:

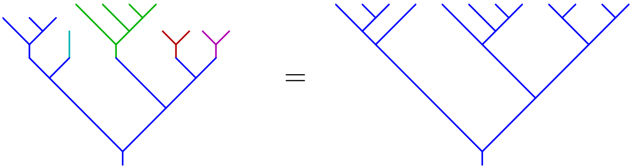
$$x^{(2)} \cdot x^{(3)} = 10x^{(5)}$$

A Hopf algebra of binary trees [LR].

Two operations on trees: Splitting



And grafting:



Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n+1$ leaves, denoted $\mathcal{Y}_n$.

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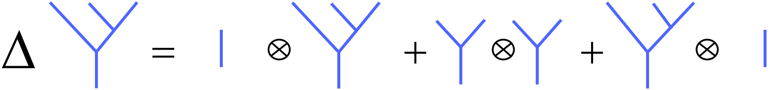

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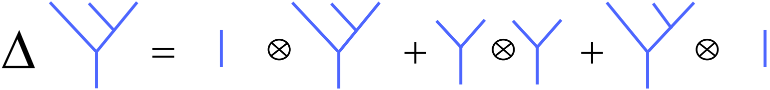
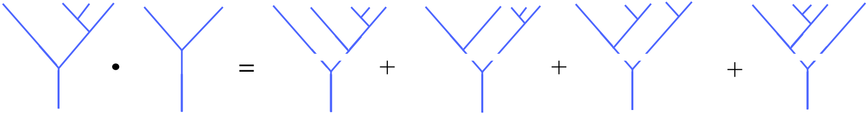


Here is how to multiply two trees:



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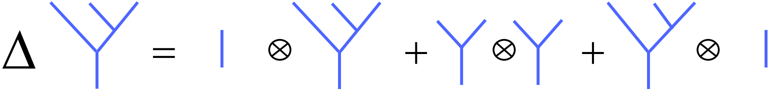

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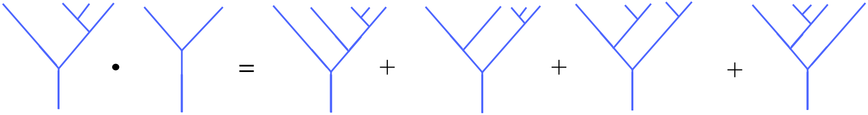
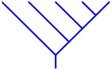


The long way to multiply in $\mathfrak{C}Sym$.

We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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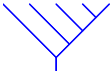
We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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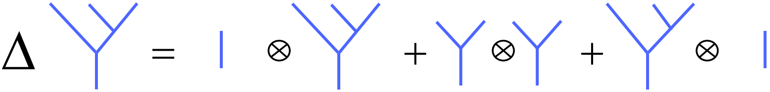

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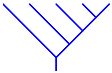
We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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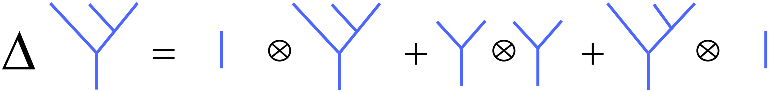

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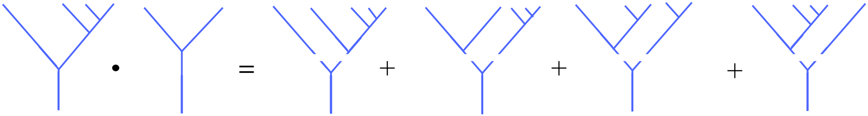

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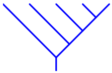
We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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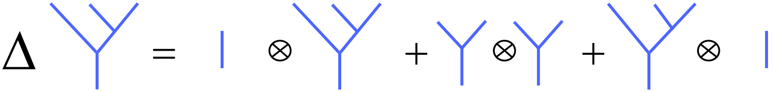

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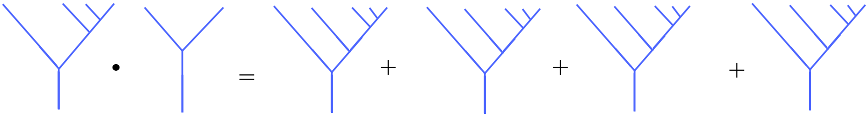


Composition of coalgebras

The idea is that given two graded coalgebras we can combine them in a way reminiscent of operad composition.
Let $\mathcal{C}$ and $\mathcal{D}$ be two graded coalgebras. We will form a new coalgebra $\mathcal{E} = \mathcal{C} \circ \mathcal{D}$ on the vector space

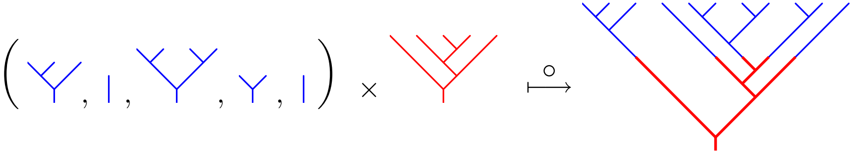
$$\mathcal{C} \circ \mathcal{D} \ := \ \bigoplus_{n \geq 0} \mathcal{C}^{\otimes (n+1)} \otimes \mathcal{D}_n \, . \tag{1}$$

Examples

The motivating example is when $\mathcal{C}$ and $\mathcal{D}$ are spaces of rooted trees. Then $\circ$ may be interpreted as some rule for grafting $n+1$ trees from $\mathcal{C}$ onto the leaves of a tree in $\mathcal{D}_n$.

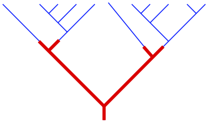
Example

Suppose $\mathcal{C} = \mathcal{D} = \mathcal{Y}Sym$ and consider some $(c_0, \ldots, c_n) \times d \in \left(\mathcal{Y}_\bullet^{\,n+1}\right) \times \mathcal{Y}_n$. Then defining $\circ$ by grafting with color coding, e.g.,

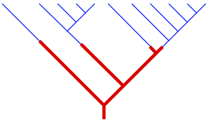


gives a new flavor of tree: *painted trees*.

A small commuting diamond



$\frac{\text{trees}}{\text{tree}}$



$\frac{\text{combs}}{\text{comb}}$

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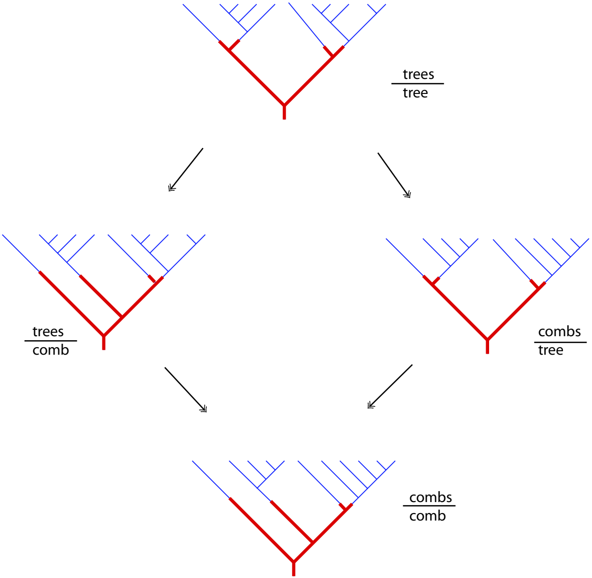


Covering coalgebras

Let $\mathcal{D}$ be a graded bialgebra, and thus a Hopf algebra.

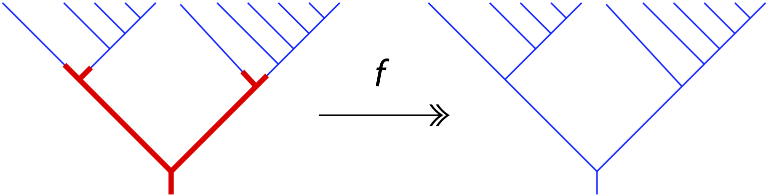
Definition

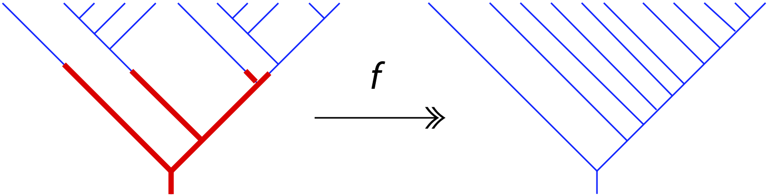
A *covering coalgebra* of $\mathcal{D}$ is a graded coalgebra $\mathcal{E}, \Delta_{\mathcal{E}}$ together with a coalgebra map $f : \mathcal{E} \to \mathcal{D}$ for which we have:

- $\mathcal{E}$ is a graded module over $\mathcal{D}$. We denote the action by $\star : \mathcal{E} \otimes \mathcal{D} \to \mathcal{E}$.
- The coalgebra map f is also a module map.
- The action and coproducts agree.

Examples

Our compositions of tree-like algebras serve as prime examples!





Covering coalgebras

A covering coalgebra turns out to give us a lot of extra free structure.

Theorem

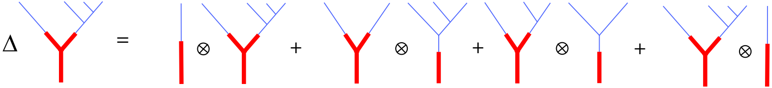
A covering coalgebra has the structures of a one-sided Hopf algebra, a Hopf module and a comodule algebra.

Proof.

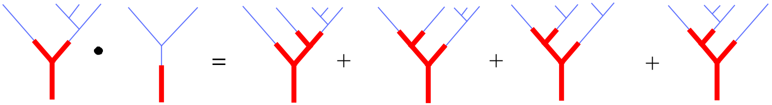
For the one-sided Hopf algebra, we define the product $m_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ by $m_{\mathcal{E}} = \star \circ (1 \otimes f)$. The one-sided unit is $1_{\mathcal{E}}$. For the other two structures we need a coaction $\rho : \mathcal{E} \to \mathcal{E} \otimes \mathcal{D}$. This coaction is defined by $\rho = (1 \otimes f) \circ \Delta_{\mathcal{E}}$ □

Examples from trees.

Here is an example of the coproduct in $\mathcal{Y}Sym \circ \mathcal{Y}Sym$:



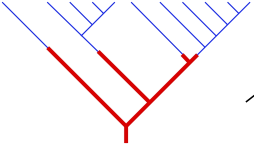
Here is an example of the product in $\mathcal{Y}Sym \circ \mathcal{Y}Sym$:



Compositions

$\mathfrak{C}Sym \circ \mathfrak{C}Sym$ is indexed by vertices of the cubes, represented by trees formed by grafting a forest of combs to the leaves of a comb (we paint the edges of the latter.)

There is a simple way to associate one of these trees with a composition.



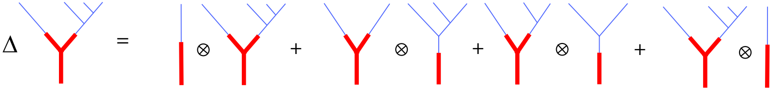
$\rightsquigarrow$ 1 4 1 5

We write the composition as the string $k_0 k_1 \dots k_j$. The number k_i is just the number of leaves of the unpainted comb grafted to the i^{th} painted edge.

Combs of combs

The coproduct is the usual splitting of trees:

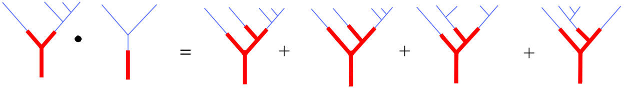
$$\Delta\ F_{13} \quad =$$



$$= \quad F_1 \otimes F_{13} \quad + F_{11} \otimes F_3 \quad + F_{12} \otimes F_2 \quad + F_{13} \otimes F_1$$

Here is the product:

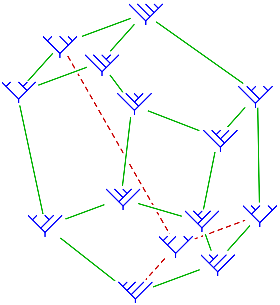
$$F_{13} \bullet F_2 \quad =$$

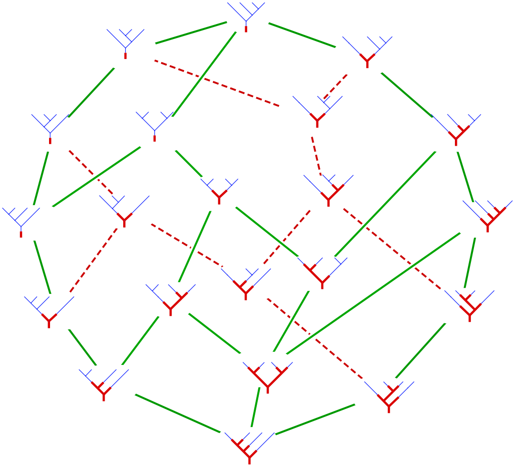


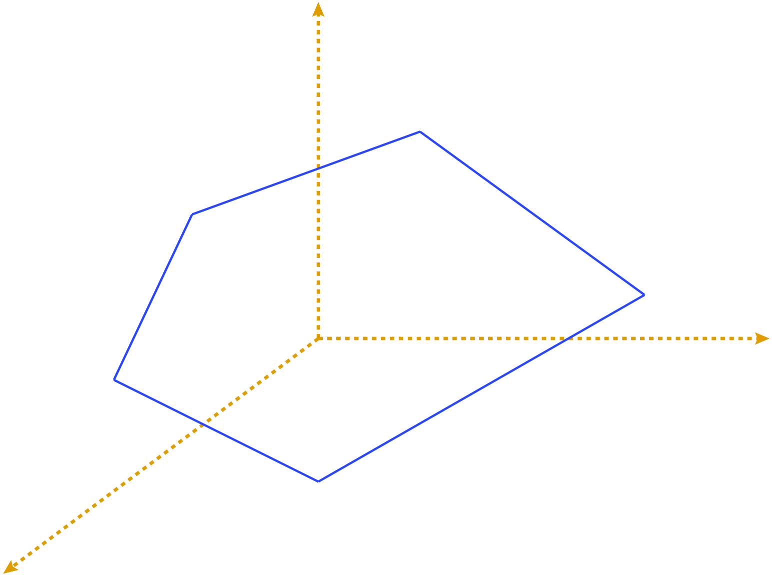
$$F_{113} \quad + F_{113} \quad + F_{122} \quad + F_{131}$$

Geometry and combinatorics.

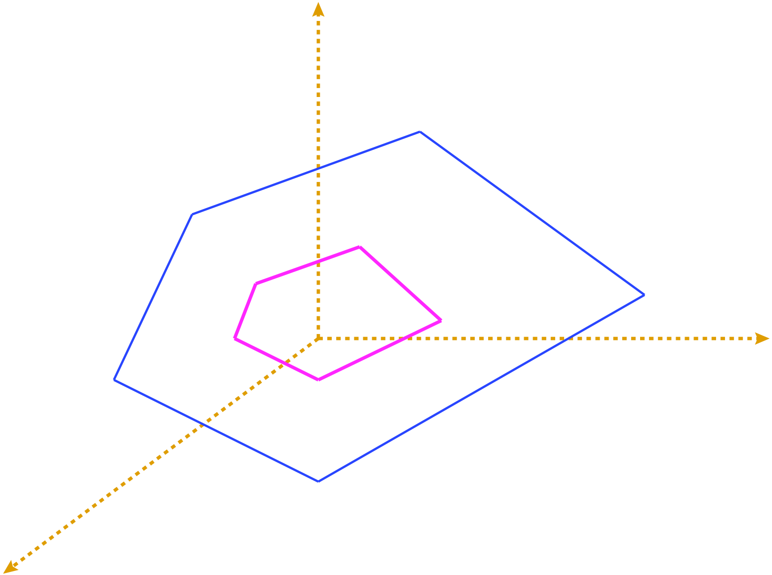
There are interesting combinatorial and geometric constructions which run in parallel to our algebraic composition of algebras.







polytopes.



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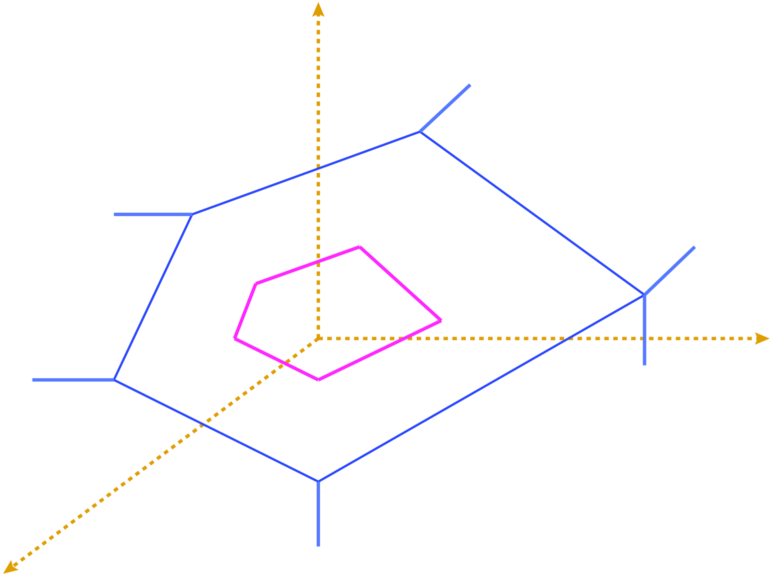

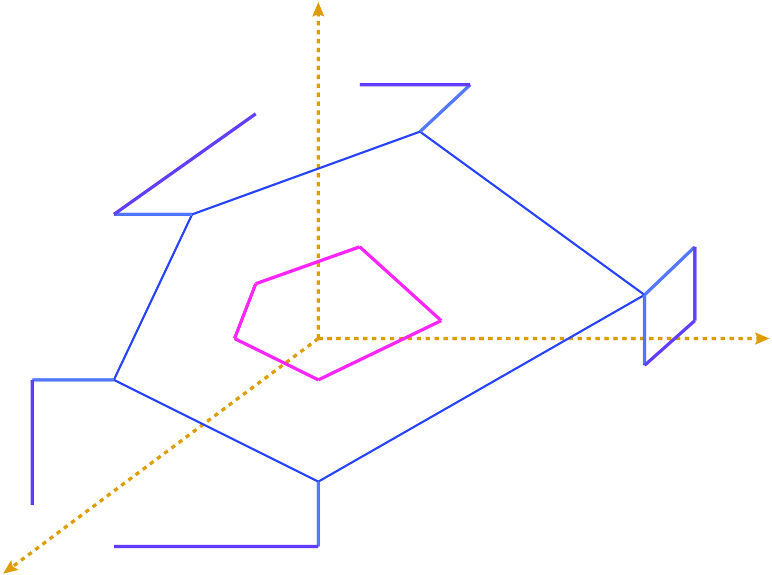

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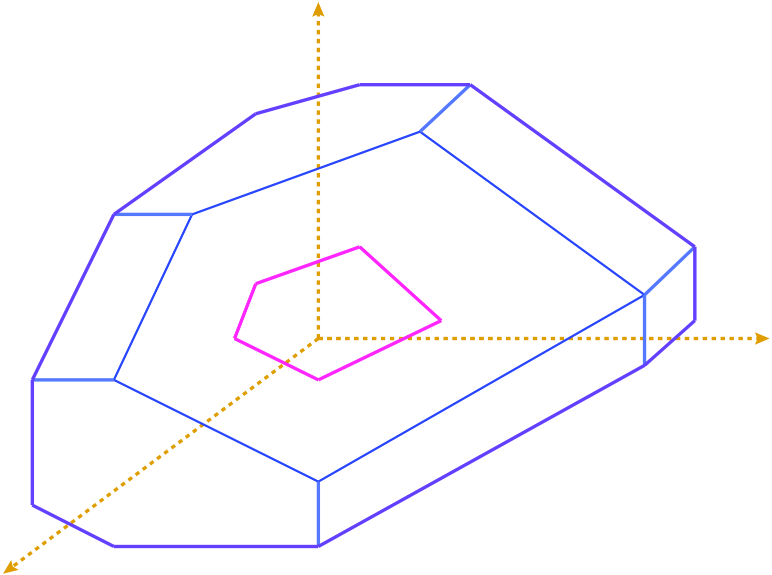

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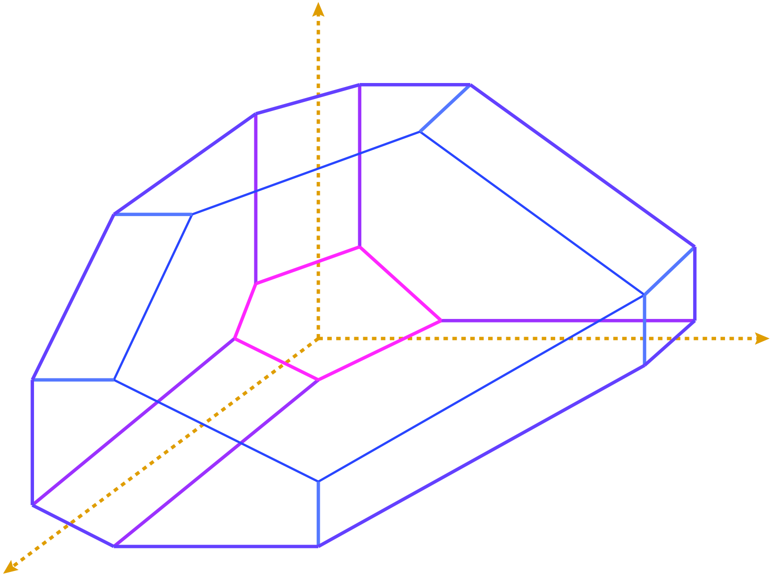

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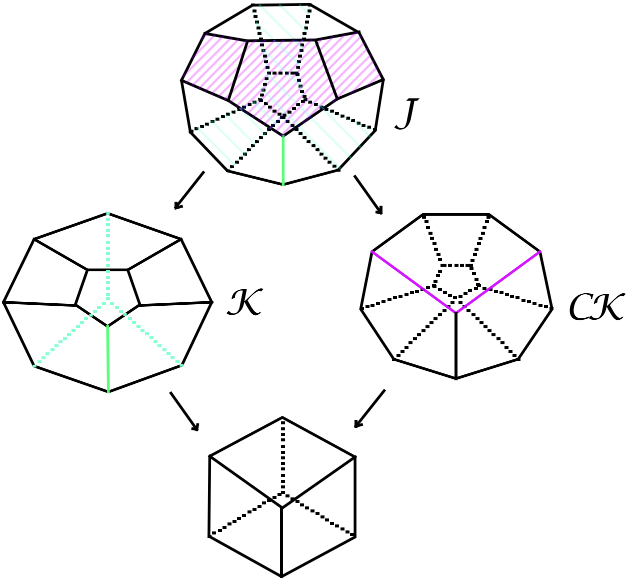
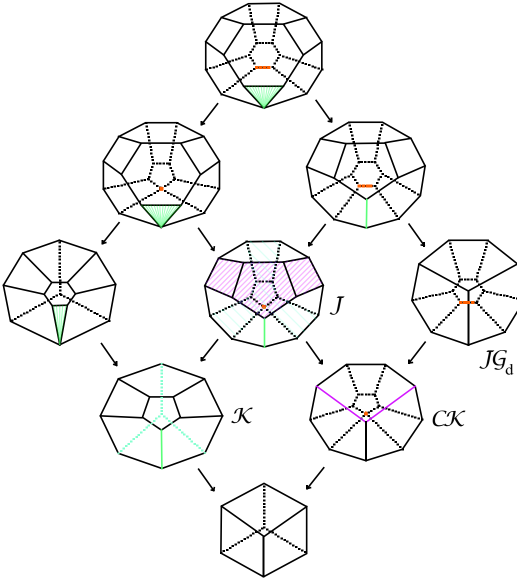


More polytopes.



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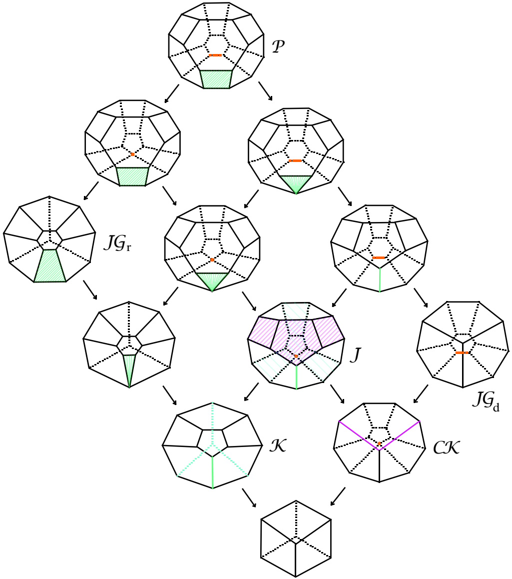

Future

More work:

• primitives and coinvariants.

•Formulas for antipodes.

Thanks!

Questions and comments?