The Hopf Algebra of Sashes

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Variation on Pattern Avoidance

Let $a \subseteq [2, k - 1]$ and $a^c$ be the elements of $[2, k - 1]$ that are not contained in $a$.

We let $y = a(k_1)a^c$, that is:

\[
\underbrace{\ldots (k_1) \ldots}_{\text{elements of } a \text{ in increasing order}} \quad \underbrace{\ldots}_{\text{elements of } a^c \text{ in increasing order}}
\]

where $y$ denotes all patterns of the form:

\[
\underbrace{\ldots (k_1) \ldots}_{\text{elements of } a \text{ in any order}} \quad \underbrace{\ldots}_{\text{elements of } a^c \text{ in any order}}
\]
Pattern Avoidance Examples

\[ y = 24(61)35 \] represents the patterns:

\[ 24(61)35, 24(61)53, 42(61)35, 42(61)53 \]

Example

573182946 does not avoid \( y \).

Define

\[ \text{Av}_n[y] = \{ x \in S_n \mid x \text{ avoids all of the patterns represented by } y \} \]

and

\[ \text{Av}_n[y_1, y_2, \ldots, y_m] = \text{Av}_n[y_1] \cap \text{Av}_n[y_2] \cap \cdots \cap \text{Av}_n[y_m]. \]
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The Hopf Subalgebra

The Malvenuto-Reutenauer Hopf algebra of permutations is given by:

\((\mathbb{K}[S_\infty], \bullet, \Delta)\)

A Hopf subalgebra of avoiders is given by:

\((\mathbb{K}[Av_\infty], \bullet_{Av}, \Delta_{Av})\)

(\(\bullet_{Av}\) and \(\Delta_{Av}\) are defined extrinsically in terms of embedding)
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A Hopf subalgebra of avoiders is given by:

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($$\cdot_{Av}$$ and $$\Delta_{Av}$$ are defined extrinsically in terms of embedding)

General Program:

- Find combinatorial objects in bijection with avoiders
- Describe product and coproduct intrinsically in terms of objects.
A particular set of Pattern Avoiding Permutations

$P_n = \{ \text{Av}_n[2(31), (41)23] \}$

<table>
<thead>
<tr>
<th>n</th>
<th>Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>12, 21</td>
</tr>
<tr>
<td>3</td>
<td>123, 132, 213, 312, 321</td>
</tr>
<tr>
<td>4</td>
<td>1234, 1243, 1324, 1423, 1432, 2134, 2143, 3124, 3214, 4213, 4312, 4321</td>
</tr>
</tbody>
</table>

- No decent larger than 2
- In each decent: $(j+2, j)$ the entry $j+1$ is to the right
Counting $P_n$

In a permutation in $P_n$, where can the $n$ be?

1. At the end: $\cdots n$
2. Before n-1: $\cdots n(n-1)\cdots$
3. Before n-2, but only if n-1 is at the end: $\cdots n(n-2)(n-1)$

Thus: $|P_n| = 2|P_{n-1}| + |P_{n-2}|$
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1. At the end: $\cdots n$
2. Before n-1: $\cdots n(n-1)\cdots$
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Thus: $|P_n| = 2|P_{n-1}| + |P_{n-2}|$

Pell Numbers:
0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, \ldots

$p_0 = 0$
$p_1 = 1$
$p_n = 2p_{n-1} + p_{n-2}$, for $n > 1$


<table>
<thead>
<tr>
<th>n</th>
<th>$\Sigma_n = {\text{Sashes of length } n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\begin{array}{c} \text{ Black square: } \blacksquare \ \text{ White square: } \square \end{array}$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{array}{c} \blacksquare \blacksquare, \square \blacksquare, \blacksquare \square, \square \square, \square \square \end{array}$</td>
</tr>
</tbody>
</table>
| 3 | $\begin{array}{c} \blacksquare \blacksquare \blacksquare, \blacksquare \blacksquare \square \blacksquare, \blacksquare \square \blacksquare \blacksquare, \square \blacksquare \blacksquare \blacksquare, \blacksquare \blacksquare \blacksquare \blacksquare, \\
\blacksquare \square \blacksquare \blacksquare, \blacksquare \square \blacksquare \blacksquare, \blacksquare \blacksquare \square \blacksquare, \blacksquare \square \square \blacksquare, \blacksquare \blacksquare \square \square \end{array}$ |

- Counting $\Sigma_n$: $|\Sigma_n| = 2|\Sigma_{n-1}| + |\Sigma_{n-2}|$
Map $\sigma : S_n \rightarrow \Sigma_{n-1}$

Example
Let $\pi = 1532476 \in S_7$: 
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Example

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2 is right of 1 $\rightarrow$ □
Example

Let $\pi = 1532476 \in S_7$:

- 2 is right of 1  \rightarrow \quad \begin{array}{c}
\end{array}

- 3 is left of 2  \rightarrow \quad \begin{array}{c}
\end{array}
Example

Let $\pi = 1532476 \in S_7$:

2 is right of 1

3 is left of 2

4 is right of 3
Map $\sigma: S_n \to \Sigma_{n-1}$

Example
Let $\pi = 1532476 \in S_7$:

2 is right of 1 $\rightarrow$  
3 is left of 2 $\rightarrow$  
4 is right of 3 $\rightarrow$  
5 is left of 4 and left of 3 $\rightarrow$  
Example

Let $\pi = 1532476 \in S_7$:

- 2 is right of 1
- 3 is left of 2
- 4 is right of 3
- 5 is left of 4 and left of 3
- 6 is right of 5
Example

Let $\pi = 1532476 \in S_7$:

- 2 is right of 1
- 3 is left of 2
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- 5 is left of 4 and left of 3
- 6 is right of 5
- 7 is left of 6
Example

Let $\pi = 1532476 \in S_7$:

- 2 is right of 1
- 3 is left of 2
- 4 is right of 3
- 5 is left of 4 and left of 3
- 6 is right of 5
- 7 is left of 6

$\sigma$ restricts to a bijection from $P_n$ to $\Sigma_{n-1}$
Graded Vector Spaces of Sashes and Pell Permutations

- **Sash** = $\bigoplus_{n \geq 0} \mathbb{K} [\Sigma_{n-1}]$
- **Pell** = $\bigoplus_{n \geq 0} \mathbb{K} [P_n]$

<table>
<thead>
<tr>
<th>Grade</th>
<th>Sashes</th>
<th>Pell Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>“∅”</td>
<td>∅ (permutation)</td>
</tr>
<tr>
<td>1</td>
<td>$\parallel$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$\Sigma_{n-1}$</td>
<td>12, 21</td>
</tr>
<tr>
<td>n</td>
<td>$\Sigma_{n-1}$</td>
<td>$P_n$</td>
</tr>
</tbody>
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Partial Order

- If $S_2 \neq S_3$, then $S_1 \sqsubseteq S_2 < S_1 \sqsubseteq S_2$
- If $S_2 = S_3$, then $S_1 \sqsubseteq S_3 < S_1 \sqsubseteq S_3 < S_1 \sqsubseteq S_3$
Partial Order

- If \( S_2 \neq S_3 \), then \( S_1 \square S_2 < S_1 \square S_2 \)
- If \( S_2 = S_3 \), then \( S_1 \square S_3 < S_1 \square S_3 \)

Example
\( \Sigma_3 \)
Example $\Sigma_4$
Product of Sashes

The product is a sum over an interval in the partial order.

Examples

\[ s \times s = [\text{rectangle}, \text{rectangle}] = [\text{rectangle} + \text{rectangle}, \text{rectangle} + \text{rectangle}] \]
Coproduct of Sashes

\[ \Delta_S(S) = \sum_{\text{dottings}} (\text{some interval}) \otimes (\text{some other interval}) \]

**Definition**
A dotting is a sash with dots alternating between \( \boxed{\bullet} \) (or \( \boxed{\circ} \)) and \( \boxed{\circ} \) (or \( \boxed{\bullet} \)), with no instances of \( \boxed{\circ\bullet} \) or \( \boxed{\bullet\circ} \).
Coproduct of Sashes

\[
\begin{align*}
\text{\ } & \quad \emptyset \otimes \begin{array}{c}
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\end{array} \begin{array}{c}
\text{\hskip 2em} \otimes \emptyset
\end{array} \\
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\end{align*}
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New framework for coproduct

In general:

\[ \Delta_{\text{Av}} = \sum_{T \subseteq [n]} I_T \otimes J_T \]

where \( I_T \) and \( J_T \) are intervals in the induced order on avoiders.
New framework for coproduct

In general:

$$\Delta_{Av} = \sum_{\substack{T \subseteq [n] \atop \text{T is good}}} I_T \otimes J_T$$

where $I_T$ and $J_T$ are intervals in the induced order on avoiders.

This is a new way of looking at the coproduct on:

- $\text{NSym} \ (Av_n[2(31),(31)2])$
- $\text{LR} \ (Av_n[2(31)])$
- diagonal rectangulations $(Av_n[2(41)3,3(41)2])$
- generic rectangulations $(Av_n[24(51)3,3(51)24])$
Consider Hopf subalgebras over vector spaces with basis elements indexed by $\text{Av}[2(31), (k1)23\ldots k−1]$

- Combinatorial object: partial evaluations with evaluated blocks of size $\leq k−2$
Generalizations

- Consider Hopf subalgebras over vector spaces with basis elements indexed by $A^v[2(31),(k1)23\cdots k−1]$
  - Combinatorial object: partial evaluations with evaluated blocks of size $\leq k−2$

\[
\begin{array}{cccccc}
\square & \square & \square & \square & \square & \square
\end{array}
\]
Generalizations

Consider Hopf subalgebras over vector spaces with basis elements indexed by $\text{Av}[2(31), (k1)23\cdots k−1]$

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\[
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Generalizations

Consider Hopf subalgebras over vector spaces with basis elements indexed by $\text{Av}[2(31), (k1)23\cdots k−1]$

Combinatorial object: partial evaluations with evaluated blocks of size $\leq k−2$

Recall: $P_n = \text{Av}[2(31), (41)23]$