Shapes and Lattices

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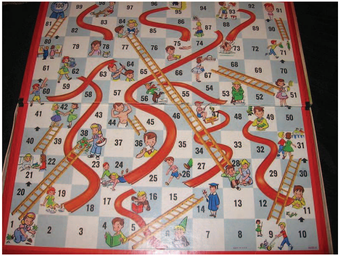


What is a lattice?

- A set,
- with a partial ordering $<$,
- such that every pair of elements has a least upper bound and a greatest lower bound.

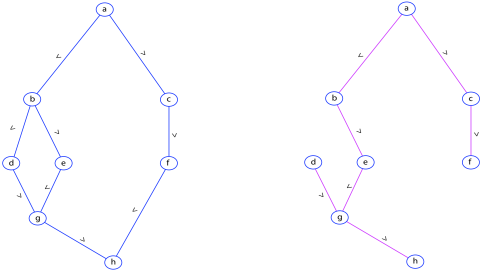
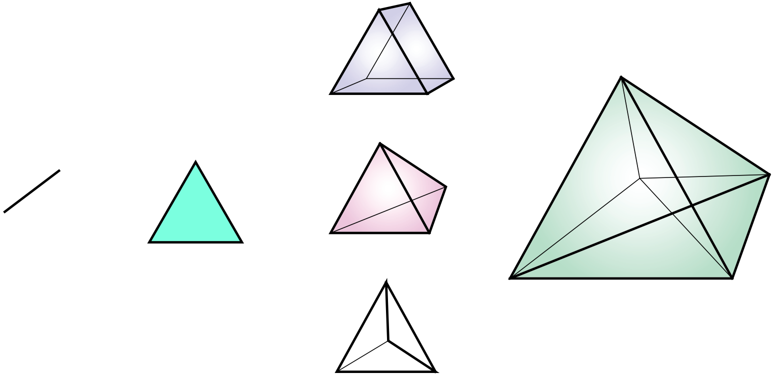


Figure: One of these is not a lattice. How about “Snakes and ladders?”

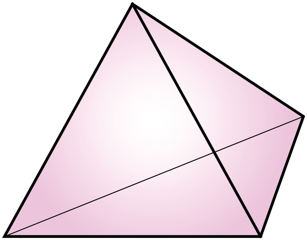
What is a polytope?

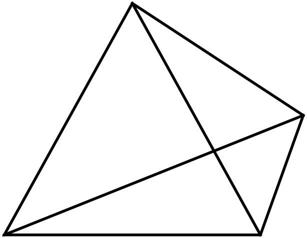
- A set of vertices in $\mathbb{R}^n$,
- and their convex hull.



What is a polytope?

- A set of vertices in $\mathbb{R}^n$,
- and their convex hull.
- The *1-skeleton of a polytope* = vertices and edges.

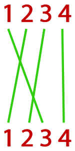




Two permutations in $\mathfrak{S}_4$.

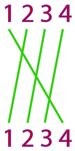
(3 1 2 4)

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{matrix}$$



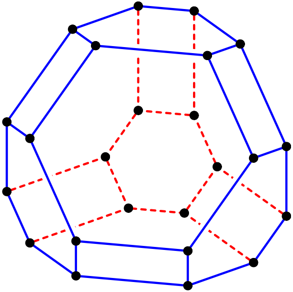
(4 1 2 3)

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{matrix}$$



Graphing permutations.

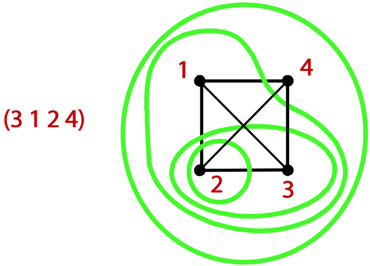
Treating the permutations as vertices and taking their convex hull yields a polytope. For example (3 1 2 4) becomes $(3, 1, 2, 4)$, and all the points from $\mathfrak{S}_4$ make this:

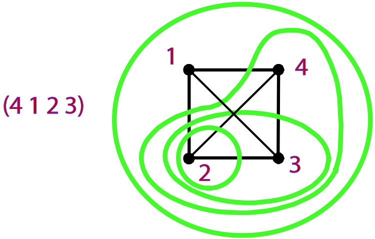


This polytope is called the permutohedron, $\mathcal{P}_n$. Why is $\mathcal{P}_4$ 3-dimensional?

Picturing permutations.

Two examples:



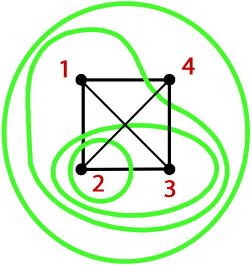


The nodes are the inputs for the permutation, and the output is the relative circle size. In the first example the image of 2 is 1, and so we put the smallest circle around 2.

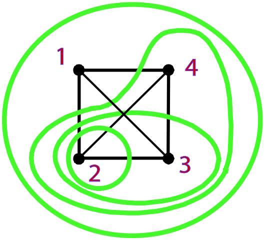
Ordering permutations.

Write down the sets of nodes in the circles: the tubes. Only one pair of tubes will differ. Compare the two numbered nodes of these which are in no smaller tubes. Here $1 < 4$.

(3 1 2 4)

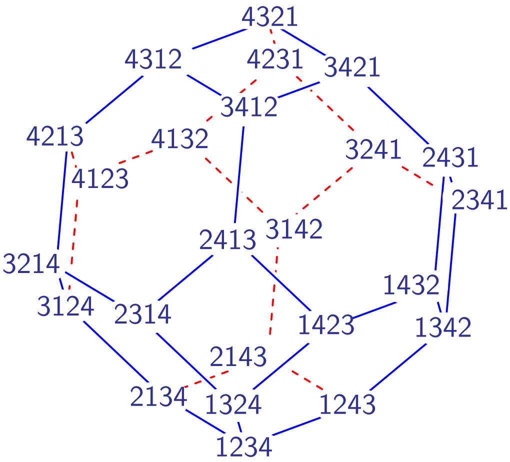


(4 1 2 3)



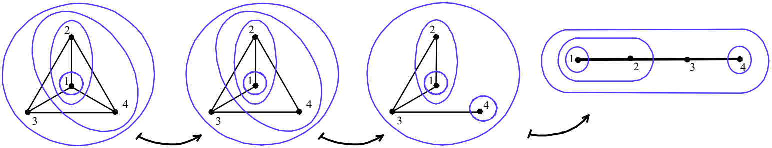
{ 2 }		{ 2 }
{ 3 2 }		{ 3 2 }
{ 3 2 1 }	$<$	{ 4 3 2 }

The 1-skeleton of $\mathcal{P}_n$ as a lattice.



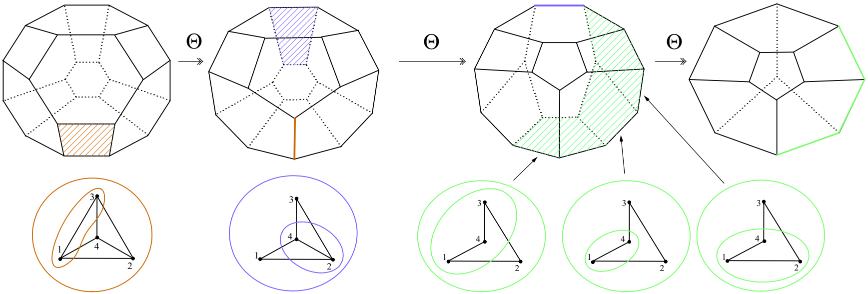
Idea.

1. Generalize permutations by deleting graph edges from the complete graph.
2. If a circle no longer surrounds a connected subgraph, split it into two.
3. Note: sometimes several permutations will be mapped to the same graph tubing.



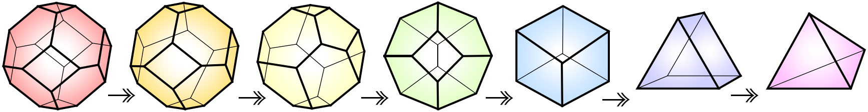
Question 1.

Is the result of deleting the same edges in all the pictures of $\mathfrak{S}_n$ still a polytope?
Yes! These are the graph associahedra, discovered by M. Carr and S. Devadoss. The edge deletions correspond to cellular projections.



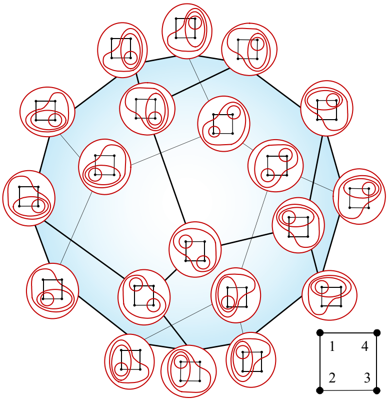
Question.

Is the 1-skeleton of each of these still a lattice?



Answer.

At least sometimes. For example, the cycle graphs: their polytopes are called the cyclohedra $\mathcal{W}_n$. Here is the lattice:

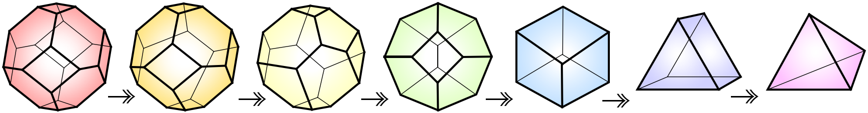


Question.

Does the projection function from $\mathcal{P}_n$ to our new lattice form a lattice congruence?

Definition: A lattice congruence is a projection of lattices that preserves least upper bounds and greatest lower bounds.

Conjecture: Yes.



Applications: cyclohedron.

1. R. Bott and C. Taubes used the space $\mathcal{W}_n \times S^1$ to define new invariants which reflect the self linking of knots.
2. S. Devadoss discovered a tiling of the $(n-1)$-torus by $(n-1)!$ copies of $\mathcal{W}_n$.
3. Recently J. Morton and collaborators used $\mathcal{W}_n$ to look for the statistical signature of periodically expressed genes in the study of biological clocks.

Thanks! For bibliography please see `http://faculty.tnstate.edu/sforcey/cyclo_alg.pdf`.