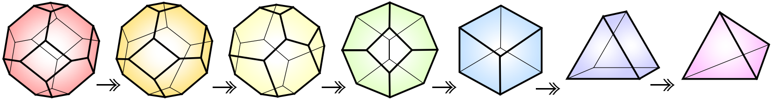
Algebras of polytopes based on network topology.

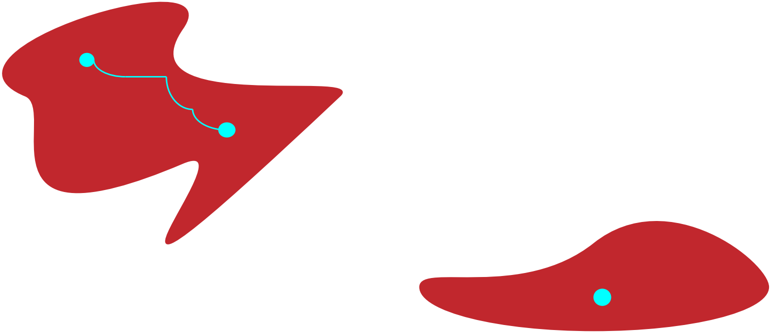
Stefan Forcey

March 23, 2010



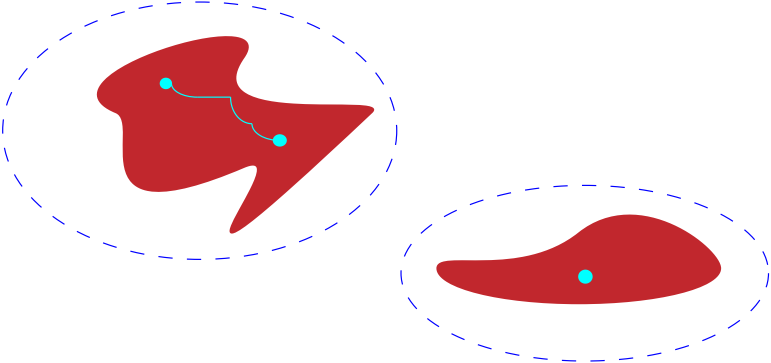
Topology

Topology is about the connectivity of space. Intuitively, two points in space are connected if there is a path we can trace between them.



Topology

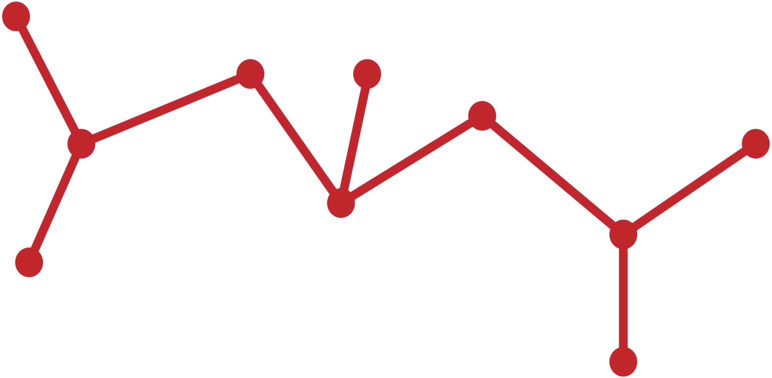
Topology is about the connectivity of space. Intuitively, two points in space are connected if there is a path we can trace between them.



More generally, they are connected if no two disjoint open sets cover the space while separating those points.

Topology and Graphs

Graphs are also about connectivity–of a network.

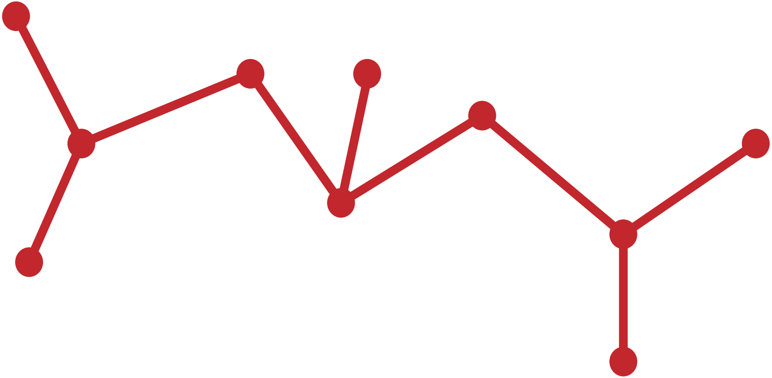


We will consider spaces whose points are the nodes of the graph

...

Topology and Graphs

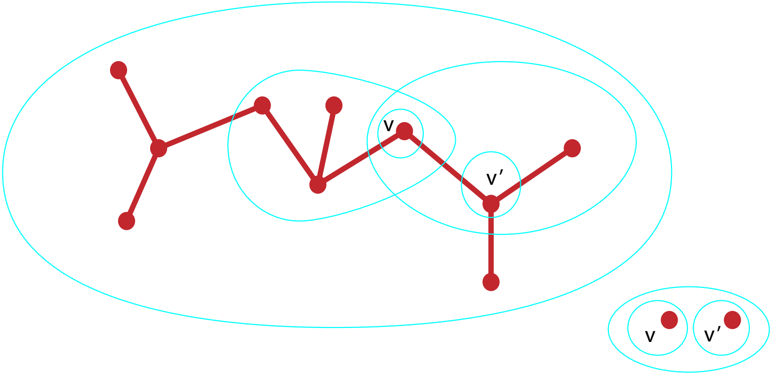
... and whose edges determine the topology.



A problem: given a graph, find several examples of topological bases such that if $\{v, v'\}$ are the endpoints of an edge then $\{v, v'\}$ is connected as a subspace.

What is a basis?

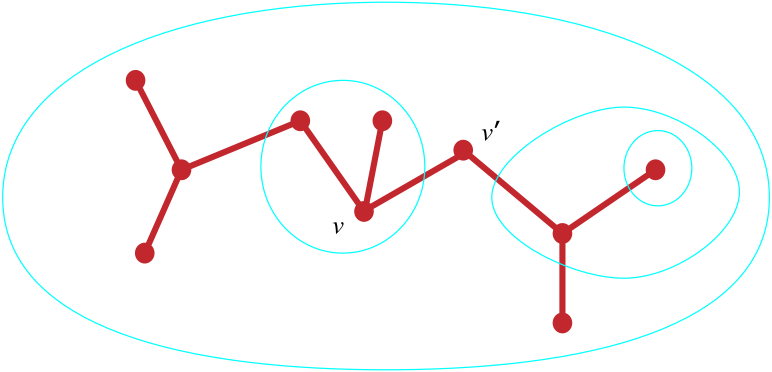
Recall that a basis is a collection of open sets which must cover both the whole space (here the set of nodes) and any of its own intersections.



A subspace topology on $\{v, v'\}$ is found by intersecting the open sets with that pair. Note that this basis fails to connect $\{v, v'\}$.

A basis which works.

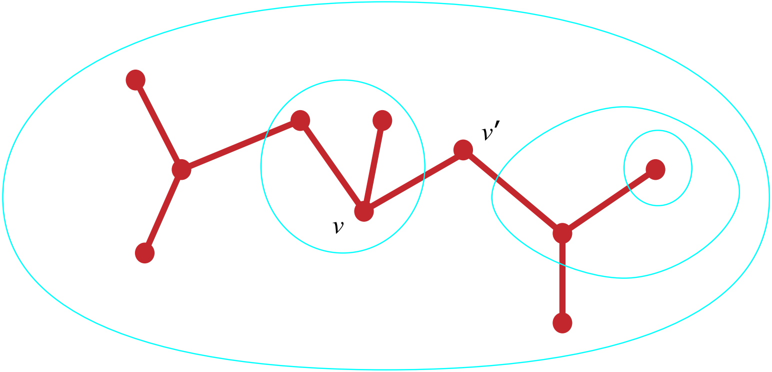
Here is an answer to the problem:



Part 2 of the problem is this: does knowing one of these bases determine the graph itself?

A basis which works.

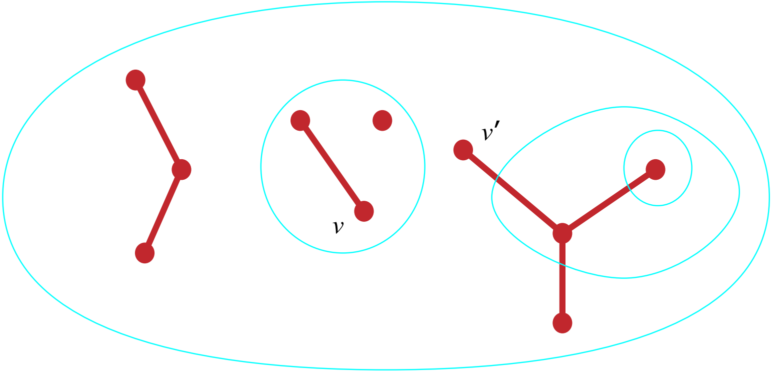
Here is an answer to the problem:



Part 2 of the problem is this: does knowing one of these bases determine the graph itself? No. Just delete an edge or two to see why.

What is a basis?

Here is an answer to the problem:



Part 2 of the problem is this: does knowing one of these bases determine the graph itself? No. Just delete an edge or two to see why.

A better question.

Given a graph G with node set V, choose a basis $\mathcal{B}$ on those nodes as follows:

Definition

1. Require that the elements of $\mathcal{B}$ induce connected subgraphs.
2. Require: if there exists $t \in \mathcal{B} - V$ such that $\{v, v'\} \cup t$ induces a connected subgraph, then $\{v, v'\}$ is connected as a subspace.

We call such a basis a *topological tubing.*

A better question.

Does knowing all these topological tubings mean that you know the graph G?

A better question.

Does knowing all these topological tubings mean that you know the graph G?

Theorem

Yes!

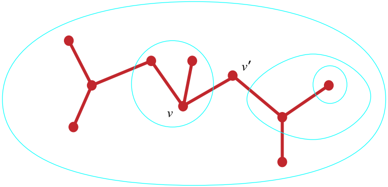
A better question.

Does knowing all these topological tubings mean that you know the graph G?

Theorem

Yes!

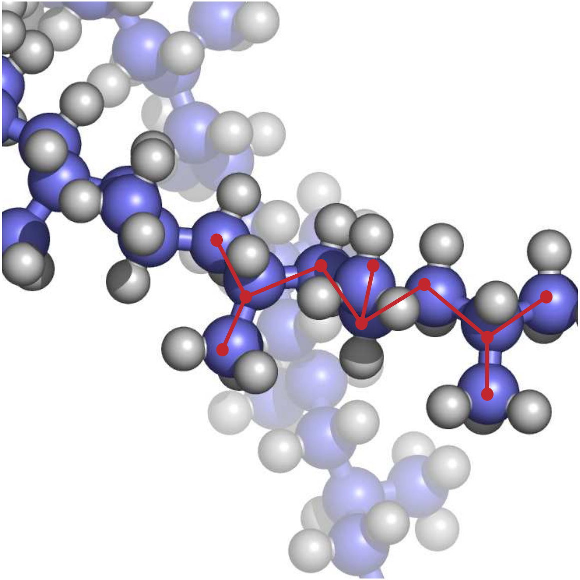
Just notice that $\{v, v'\}$ is an edge if and only if $\{v, v'\}$ is connected for all $\mathcal{B}$.



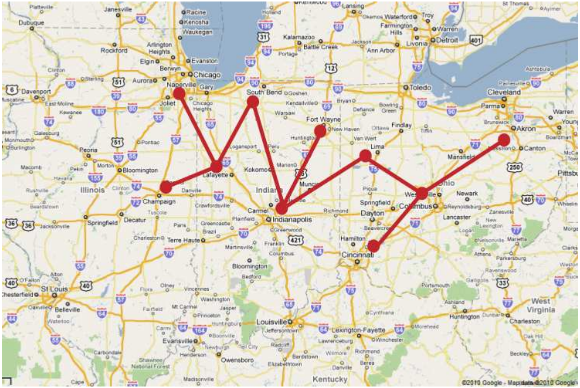
However, a G-tubing has a little more in its definition than we need. So we digress to talk about the meaningfulness of such a model. First, some examples of the ubiquity of networks:

polypropylene

underlying picture source: wikimedia



underlying picture source: google maps



power grid

underlying picture source: google street view

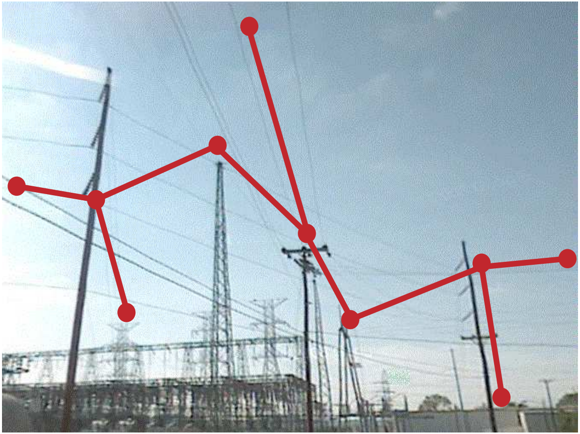
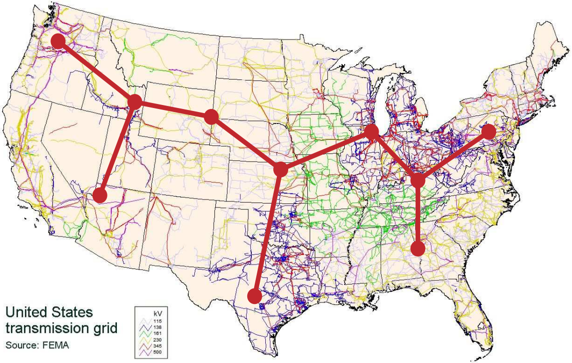


Figure: A substation near my house.

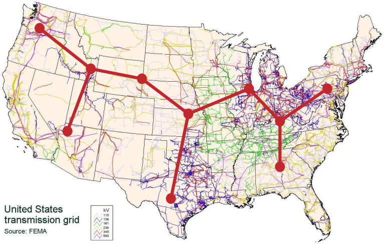
power grid

underlying picture source: wikimedia



Grid topology.

In a power grid, we might want to identify families of subnetworks which are “locally insulated.”



By this we mean that if a given sub-grid t fails, then the only other sub-grids directly affected (via a direct connection) are those which contain that entire sub-grid t.

Theorem 2.

The second theorem surprisingly says that this local insulation holds if and only if the family of subsets of nodes is a toplogical tubing.

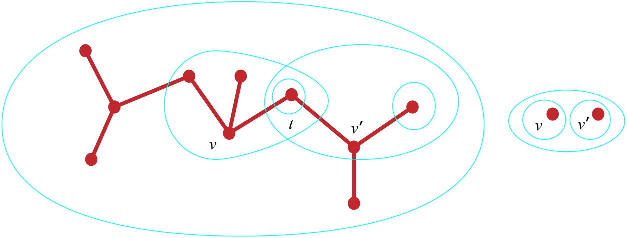
Theorem

A topological tubing as we define it here is the same as a combinatorial graph tubing.

The latter is defined by M. Carr and S. Devadoss, in *Coxeter Complexes and Graph-Associahedra*. Their definition of a tubing is that the induced connected subgraphs, called tubes, must either be nested or far apart (not connected by an edge.) This is precisely our local insulation!

Proof idea and Questions.

Part of the proof, by example:



Note that this basis will not obey the second requirement.

Q_1. How are the tubings of G related to each other, topologically and combinatorially?

Q_2. What sort of algebraic structures do they support?

Quick answers.

Q_1. How are the tubings related, topologically and combinatorially?

A_1 They are the faces of polytopes!

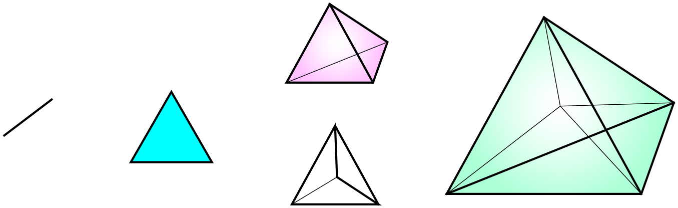
Q_2. What sort of algebraic structures do they support?

A_2 Free vector spaces on certain sequences of tubings are graded Hopf algebras!

What is a polytope?

- A set of vertices in $\mathbb{R}^n$,
- and their convex hull.

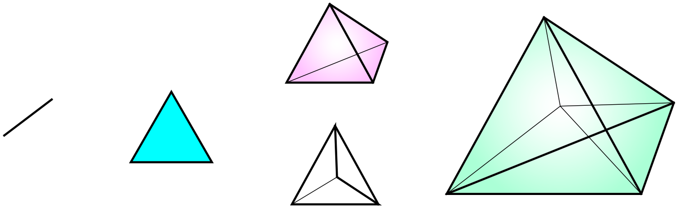
Recall that a 1 dim. polytope is a line segment, a 2 dim. polytope is a polygon, a 3 dim. polytope is a polyhedron and that 4 dim. polytopes are hard to draw.



What is a polytope?

- A set of vertices in $\mathbb{R}^n$,
- and their convex hull.

Recall that a 1 dim. polytope is a line segment, a 2 dim. polytope is a polygon, a 3 dim. polytope is a polyhedron and that 4 dim. polytopes are hard to draw.



What is a Hopf algebra?

- A vector space W,
- with a product $W \otimes W \longrightarrow W$,
- and a coproduct $\Delta : W \longrightarrow W \otimes W$,
- with unit, counit and an antipode map.

The coproduct is an algebra homomorphism.

What is a Hopf algebra?

- A vector space W,
- with a product $W \otimes W \to W$,
- and a coproduct $\Delta : W \to W \otimes W$,
- with unit, counit and an antipode map.

The coproduct is an algebra homomorphism.

What is a Hopf algebra?

- A vector space W,
- with a product $W \otimes W \to W$,
- and a coproduct $\Delta : W \to W \otimes W$,
- with unit, counit and an antipode map.

The coproduct is an algebra homomorphism.

What is a Hopf algebra?

- A vector space W,
- with a product $W \otimes W \to W$,
- and a coproduct $\Delta : W \to W \otimes W$,
- with unit, counit and an antipode map.

The coproduct is an algebra homomorphism.

Long answers.

A_1. Devadoss and Carr discovered the main relationship between all the tubings of a given graph. When you relate them by containment (of tubes and of topologies), they form a polytope. A subface is a basis for a finer topology than the face it is in.

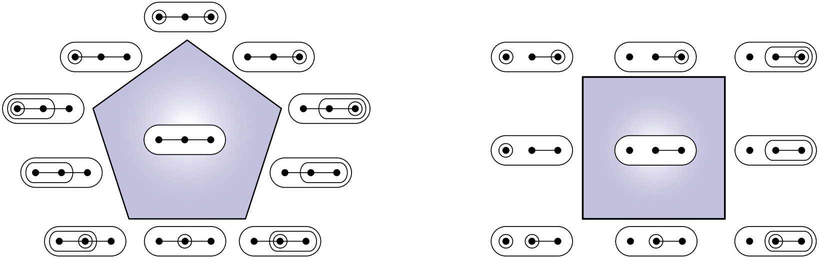
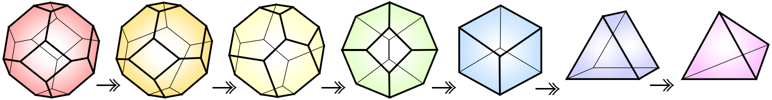


Figure: Graph associahedra of a path and a disconnected graph.

3d examples.

Here are some graph associahedra:



This string includes 3d examples of the permutohedra from complete graphs, cyclohedra from cycle graphs, associahedra from paths, and simplices from edgeless graphs.

Building polytopes.

The tubings which are single tubes make up the facets of the polytope. For examples, we will actually look at some multigraphs. The definition of combinatorial tubing is the same.

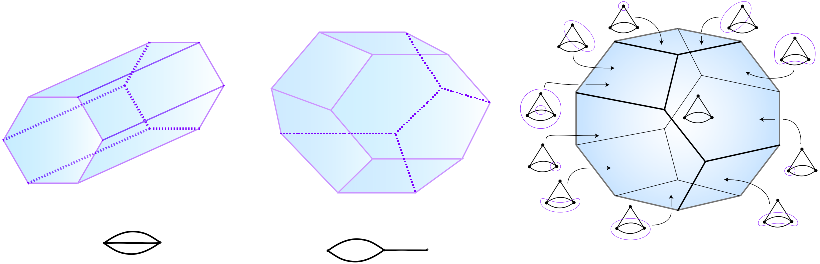


Figure: Several multigraph associahedra. The third is shown with labeled facets, and we will reconstruct the others on the white board.

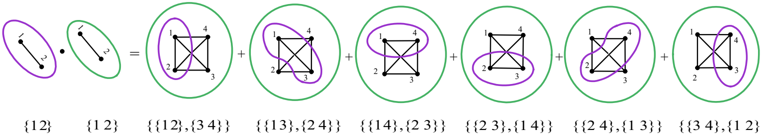
More long answers.

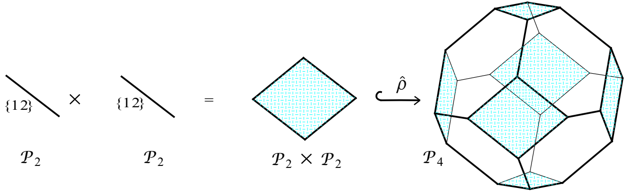
A_2. The vertices of the permutohedra and the associahedra form the algebraic bases for famous graded Hopf algebras. [Malvenuto-Reutenauer] and [Loday-Ronco]. The vertices of the n-dimensional polytope make up the n^{th} component of the basis.

Chapoton discovered that actually the entire sets of faces for the n-dimensional polytopes in these families actually support a graded Hopf algebra.

Example.

Here is how to multiply tubings of the complete graph to get faces of the permutohedron:

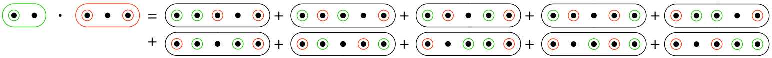




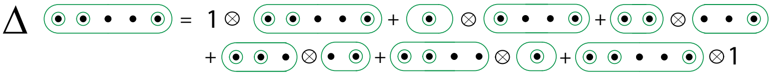
Note that the operands are subspaces of the product, in all possible ways.

There is much ongoing work generalizing these to other classes of graphs. Recently we found a Hopf algebra for the simplices, an algebra for the cycle graphs, and together with M. Ronco we are looking for more.

Here is how to multiply some tubings of simplices:



...and here is the coproduct:



Future directions and open questions.

- A famous open question: how many distinct topologies are there on n elements?
- An open question: does the graph associahedron completely determine the graph?
- What are the generating functions for counting multigraph tubings?
- What is the topological interpretation of a tubing on a multigraph?
- What is the Hopf algebraic structure of the geometric combinatorics of the topological structure of pseudographs?