Examples:

1) How many ways can 15 wrestlers (same class) place in the 2021 Olympics: gold, silver, bronze?

\[ \binom{15}{3} = \frac{15!}{12!3!} = 2730 \]

2) How many ways can 40 students fill up the front row of 5 seats (in order)?

\[ 40^5 = 40 \times 39 \times 38 \times 37 \times 36 = 78,960,960 \]

Next we return to Addition and Subtraction.

In terms of sets:

Ex: \( A = \{1, 2, 5\} \), \( B = \{3, 7\} \), \( U = \{1, \ldots, 7\} \)

\[
\begin{array}{c|c|c}
& 1 & 2 \\
\hline
\hline
A & 1 & \times \\
\hline
\hline
B & 5 & \times \\
\hline
\hline
U & 3 & \times
\end{array}
\]

When \( A \cap B = \emptyset \), then \( |A \cup B| = |A| + |B| = 5 \)

When \( U \) is the universal set, \( |\bar{A}| = |U| - |A| = 4 \)

When \( A \cap B = \emptyset \), \( |\overline{A \cup B}| = |U| - |A| - |B| = 2 \)
When there is overlap we have to avoid over counting!

Ex \( A = \{1, 2, 3, 7\} \quad B = \{4, 7\}\)

\[
|A \cup B| = |A - B| + |A \cap B| + |B - A| = 3 + 1 + 1 = 5
\]

or, shortcut:

\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

= 4 + 2 - 1 = 5

\[
\text{in in the shared stuff, \{7\}, get counted twice: once in A, once in B, so we subtract it!}
\]

And \( |A \cup B| = |U| - |A| - |B| + |A \cap B| = 7 - 4 - 2 + 1 = 2\)

For 3 or more sets we can alternately add and subtract the overlaps to get the total:

Ex \( A = \{1, 2, 3, 7\} \quad B = \{6, 7, 4\} \quad C = \{2, 3, 4, 7\}\)

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|
\]

= 4 + 3 + 4 - 1 - 2 - 3 + 1 = 6
and

\[ |\overline{A\cup B\cup C} | = |U| - |A| - |B| - |C| + |A\cap B| + |A\cap C| + |B\cap C| - |A\cap B\cap C| \]

\[ = 7 - 6 = 1 \]

Example

How many PINs with 5 digits
but:
  • no repeated digits
  • First digit cannot be 0
  • Third digit cannot be 2
  • Fifth digit cannot be 5

Idea: let U be all the 5 digit PINs
with no repeated digits

\[ |U| = 10^5 \]

let A be PINs with first digit 0.
let B be PINs with third digit 2.
let C be PINs with fifth digit 5.

Then legal PINs are \( \overline{A\cup B\cup C} \).