Symmetric Traveling Salesman

Polytope and problem
1) Problem

- Given: a set of cities and the symmetric distances (or costs) between each pair.
- A tour visits each city once and returns to the start.
- Find the tour with minimal total distance (cost).
- We can safely assume that the distances form a metric, obeying the triangle inequality.
2) Model and 3) Example

• The $n$ cities and distances are modeled as a complete graph $K_n$ with weighted edges.
• A tour is a Hamiltonian cycle $c$, and its total distance $p$ is the sum of its weights.

Example $n=5$:

An example tour is $c=1,3,4,2,5,1$. It’s total distance is $p = 6.3$. 
4) Polytope

• Our vertices $x(c)$ correspond to Hamiltonian cycles $c$ of the graph $K_n$. They have $\binom{n}{2}$ components, one for each edge.

• The component for edge $\{i,j\}$ is $x_{ij}=1$ if the edge is in the cycle; $x_{ij}=0$ if not.

• Our objective function $p=d \cdot x(c)$ outputs total distance. The vector $d$ has components the given edge weights.
5) Example

\[ d = <1.1, 1.1, 2, 1, 2.2, 2.1, 1.1, 1, 2, 1> \]

**Vertices:**

\[
\begin{array}{ccc}
\text{c} & \text{x(c)} & p = d \cdot x(c) \\
1,2,3,4,5,1 & <1,0,0,1,1,0,0,1,0,1> & p = 6.3 \\
1,2,3,5,4,1 & <1,0,1,0,1,0,0,0,1,1> & p = 8.3 \\
1,2,4,3,5,1 & & \\
1,2,4,5,3,1 & & \\
1,2,5,3,4,1 & & \\
1,2,5,4,3,1 & & \\
1,3,2,4,5,1 & & \\
1,3,2,5,4,1 & & \\
1,4,2,3,5,1 & & \\
1,4,2,5,3,1 & & \\
1,5,2,3,4,1 & & \\
1,5,2,4,3,1 & & \\
\end{array}
\]

Etc…. 
6) Known Facts

Dimensions:
0, 2, 5, 9, 14 ... \(n(n-3)/2\)

- Numbers of vertices in \(n\)th polytope:
1, 3, 12, 60, ... \((n-1)!/2\)

- Numbers of Facets:
0, 3, 20 ,100, 3437, 194187, 42104442, ...

Source:
7) Greedy Algorithm

• Pick a starting city; choose the edge with least weight; repeat (don’t return to any city twice until the last step.)

The greedy algorithm starting at node 1 gives the cycle $c = 1,5,4,3,2,1$ with total distance $p=6.3$. 
Minimize \( p = 1.1x_{12} + 1.1x_{13} + 2x_{14} + 1x_{15} + 2.2x_{23} + 2.1x_{24} + 1.1x_{25} + 1x_{34} + 2x_{35} + 1x_{45} \) subject to

\[
\begin{align*}
  x_{12} + x_{13} + x_{14} + x_{15} & = 2 \\
  x_{12} + x_{23} + x_{24} + x_{25} & = 2 \\
  x_{23} + x_{13} + x_{34} + x_{35} & = 2 \\
  x_{24} + x_{34} + x_{14} + x_{45} & = 2 \\
  x_{15} + x_{25} + x_{35} + x_{45} & = 2 \\
  x_{12} & \geq 0 \\
  x_{13} & \geq 0 \\
  x_{14} & \geq 0 \\
  x_{15} & \geq 0 \\
  x_{23} & \geq 0 \\
  x_{24} & \geq 0 \\
  x_{25} & \geq 0 \\
  x_{34} & \geq 0 \\
  x_{35} & \geq 0 \\
  x_{45} & \geq 0 \\
  x_{12} & \leq 1 \\
  x_{13} & \leq 1 \\
  x_{14} & \leq 1 \\
  x_{15} & \leq 1 \\
  x_{23} & \leq 1 \\
  x_{24} & \leq 1 \\
  x_{25} & \leq 1 \\
  x_{34} & \leq 1 \\
  x_{35} & \leq 1 \\
  x_{45} & \leq 1
\end{align*}
\]

Optimal Solution: \( p = 5.3; \) \( x_{12} = 1, x_{13} = 1, x_{14} = 0, x_{15} = 0, x_{23} = 0, x_{24} = 0, x_{25} = 1, x_{34} = 1, x_{35} = 0, x_{45} = 1 \)

[That’s the vertex vector \( \mathbf{x}(c) = <1,1,0,0,0,1,1,0,1> \) which is the cycle \( c=1,2,5,4,3,1 \)]

[Note: You can cut and paste the above into the link given at the top and see the steps of the simplex method with the Dantzig rule.]