Suppose we are given the following data:

1. A 2-fold monoidal category $\mathcal{C}$, with tensor products $\otimes_1$ and $\otimes_2$ and interchange maps $\eta_{abcd}$: $(a \otimes_2 b) \otimes_1 (c \otimes_2 d) \to (a \otimes_1 c) \otimes_2 (b \otimes_2 d)$ satisfying the axioms described in [BF],

2. A (2-fold monoidal) natural transformation $\Lambda$ of the identity functor $1_\mathcal{C}$, and

3. A “dimension function” $\sigma: \text{Obj}(\mathcal{C}) \to \mathbb{N}$ which is additive over the tensor products: $\sigma(a \otimes_1 b) = \sigma(a) + \sigma(b)$ and $\sigma(a \otimes_2 b) = \sigma(a) + \sigma(b)$.

We propose the following construction of a related 2-fold monoidal category:

**Definition.** The category $\mathcal{C}: \Lambda$ is constructed as follows:

1. $\text{Obj}(\mathcal{C}: \Lambda)$ is the same as $\text{Obj}(\mathcal{C})$.

2. $\text{Hom}$ sets in $\mathcal{C}: \Lambda$ are the same as in $\mathcal{C}$, but restricted to morphisms between objects of the same dimension. That is,

   $$\text{Hom}_{\mathcal{C}: \Lambda}(a, b) = \begin{cases} 
   \text{Hom}_{\mathcal{C}}(a, b), & \sigma(a) = \sigma(b) \\
   \emptyset, & \text{otherwise}
   \end{cases}$$

3. $\mathcal{C}: \Lambda$ has the same tensor products $\otimes_1$ and $\otimes_2$ as $\mathcal{C}$.

4. The interchange map for $\mathcal{C}: \Lambda$ is $(\eta: \Lambda)$ defined by

   $$(\eta: \Lambda)_{abcd} = (1_a \otimes_1 \Lambda_c^\sigma(b) \otimes_2 \Lambda_b^\sigma(c) \otimes_1 1_d) \circ \eta_{abcd}$$

   where $\Lambda_x^y$ indicates $y$-fold composition of the endomorphism $\Lambda_x : x \to x$, and by convention $\Lambda_x^0$ indicates the identity map $1_x$.

**Example.**

Let $\mathcal{C}$ be the category of free $\mathbb{Z}$-modules with direct sum playing the role of both “products”, $\sigma$ equal to the rank, and the standard symmetric braiding isomorphism playing the role of the interchange map. Let $\Lambda$ be multiplication by a nontrivial scalar $x$. The twisted interchange map from $a \oplus b \oplus c \oplus d$ to $a \oplus c \oplus b \oplus d$ can be viewed as a block matrix:

$$
\begin{pmatrix}
1 & x^{\sigma(b)} \\
1 & x^{\sigma(c)} & x^{\sigma(b)} \\
1 & 1 & x^{\sigma(b)} & x^{\sigma(c)} & x^{\sigma(b)} \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

where each entry represents a scalar of the appropriate dimension. Note that if $x$ is not a unit then the twisted interchange is not an isomorphism.

The point, of course, is the following:

**Proposition.** The category $\mathcal{C}: \Lambda$ defined above satisfies the axioms of a 2-fold monoidal category.
Proof.

This requires nothing more than walking through the axioms in definition 1.7 of [BF], which are all routine. Remarks:

1. Naturality of \( \eta: \Lambda \) is straightforward but this is the reason for restricting to maps between same-dimension objects.
2. The internal/external unit conditions are satisfied due to the fact that \( \sigma(1) \) must be 0.
3. Since we are constructing a 2-fold category there is no giant hexagon to worry about.
4. The interesting part is the associativity constraints. The two legs of the internal associativity diagram can be reduced to

\[
(1_u \otimes \Lambda^{\sigma(v)}_w \otimes \Lambda^{\sigma(v)+\sigma(x)}_y \otimes \Lambda^{\sigma(w)+\sigma(x)}_v \otimes \Lambda^{\sigma(y)}_x \otimes 1_z) \circ \eta_{uw,vx,y,z} \circ \eta_{u,v,w,x}
\]

and

\[
(1_u \otimes \Lambda^{\sigma(v)}_w \otimes \Lambda^{\sigma(v)+\sigma(x)}_y \otimes \Lambda^{\sigma(w)+\sigma(x)}_v \otimes \Lambda^{\sigma(y)}_x \otimes 1_z) \circ \eta_{u,v,wy,xz} \circ \eta_{w,x,y,z}
\]

respectively (subscripts on the tensors are suppressed). Equality then follows by the internal associativity of the original \( \eta \). This is where we need the additivity of \( \sigma \) over \( \otimes_1 \), and the fact that \( \Lambda \) is a monoidal natural transformation.

5. The external associativity axiom is entirely similar and makes use of the additivity of \( \sigma \) over \( \otimes_2 \). □