1. Consider the conditional $E = "p \rightarrow q \land \sim r"$.

Use de Morgan’s laws to write simplified versions of the following :

- The *negation* of $E$ :

- The *inverse* of $E$ :

- The *converse* of $E$ :

- The *contrapositive* of $E$ :

2. Fill in the truth table below.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>$p \lor q \rightarrow \sim r$</th>
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<tbody>
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3. Determine if the following argument is valid. Justify your answer.

$p \lor q \rightarrow r$

$\sim p \land \sim q$

Therefore, $\sim r$

4. Write a *boolean expression* which will generate the following *Karnaugh map* :

<table>
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<tr>
<th>$z \land xy$</th>
<th>11</th>
<th>10</th>
<th>00</th>
<th>01</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
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</tr>
</tbody>
</table>
5. (a) Convert 110 1000 1101₂ to decimal.

(b) The value of 171₈₁₀ in binary is 110 101ₓᵧ₁₁₀₂. Find the values of the unknown digits \( x \) and \( y \).

(c) Convert 1AF9₁₆ to binary.

(d) The value of an integer \( x \) in hexadecimal is FA82BC₁₆. What is the value of \( 256x + 37 \) in hexadecimal? (Hint: \( 256 = 16² \)).

6. What is the truth set for the predicate \( P(n) = "n³ < 100" \), with domain the set of natural numbers, \( \mathbb{N} \)?

7. Consider the statement "All prime numbers are odd".

(a) Using the domain \( D = \mathbb{Z} \), and predicates \( P(x) = "x \text{ is prime}" \), and \( O(y) = "y \text{ is odd}" \), write this statement symbolically.

(b) Write a simplified symbolic negation of your statement.

(c) Express your negation in English.

8. Draw a diagram of validity for the following syllogism, and determine if the argument is valid. If so, state whether it is direct (Modus Ponens), indirect (Modus Tollens), or other. If it is not valid, describe the error.
D = all students

"All math majors are good at logic”.
"Steve is good at logic”.
"Therefore, Steve is a math major”.

Test 2

1. Consider the statement "The integers are closed under division”.
   (a) Write this statement as a symbolic statement with quantifiers.

2. (a) Find the values of $-1737 \div 15$ and $-1737 \mod 15$.
   (b) Directly prove the following :
   $(\forall a, b \in \mathbb{Z} ) (a \mod 3 = 1 \land b \mod 3 = 2 \rightarrow 3 \mid a + b)$

3. Use the Euclidean algorithm to evaluate $gcd(88725, 31977)$.

4. (a) For real numbers $x, y$, consider the two statements :
   (A) $\lfloor x \rfloor + \lceil y \rceil \geq x + y$
   (B) $\lfloor x \rfloor + \lceil y \rceil \leq x + y$.
   Which of these statements is always true? (A), (B), both, or neither?
   Carefully explain your answer.
(b) Find all real \( x \) such that \([3 - 2x] = 5\). 

5 points

5. Prove by contradiction that \( 2 + 3\sqrt{2} \) is irrational.

10 points

6. (a) Completely state (symbolically), the Principle of Mathematical Induction (PMI)

5 points

(b) Use the PMI to prove that \( 1 + 3 + \cdots (2n - 1) = n^2 \) for every positive integer \( n \).

10 points

7. (a) Find an explicit formula which generates the sequence \( \frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \cdots \).

5 points

(b) Write \( 1 - 4 + 9 - 16 + 25 - 36 \) in summation form.

5 points

(c) Evaluate and simplify the product \( \prod_{j=1}^{6} \frac{j}{j + 2} \)

5 points

(d) Suppose that \( \prod_{k=1}^{200} a_k = \frac{22}{7} \), and that \( a_{200} = 5 \).

Find the value of \( \prod_{k=1}^{199} a_k \).

5 points

(e) Using the summation formula \( \sum_{k=1}^{n} 4k^3 = n^2(n + 1)^2 \), find the value of \( \sum_{k=12}^{37} 4k^3 \).

5 points
1. Mark each of the following statements True or False, as appropriate:

(a) \{a\} \in \{a, b\}  
(b) \{a\} \subseteq \{a, b\}  
(c) If \(A, B\) are sets, then \(A - B = A - (A \cap B)\)  
(d) If \(A, B, C\) are sets, \(A \subseteq B, C \subseteq B\), then \(A \subseteq C\).  
(e) Let \(\mathcal{P}(A)\) denote the power set of the set \(A\). 
   If \(a \in A\) and \(b \in A\), then \(\{a, b\} \in \mathcal{P}(A)\).  

2. Draw the most general Venn diagram for sets \(A, B, C\), and shade the region which corresponds to \((A \cap B^c) \cup C\).

3. Consider the following claim for non-empty sets \(A, B, C\) :
   If \(A \subseteq B\) and \(C \subseteq B\), then \(A \cap C \neq \emptyset\).
   Find a counterexample to this statement.

4. Consider the following true statement :
   For any elements \(x, y\) in a Boolean algebra : \((x + y) + x.y = 1\)

   (a) Fill in the justification for the following steps of the proof : 
   \[(x + y) + x.y = ((x + y) + x).((x + y) + y)\] 
   \[= ((y + x) + x).((x + y) + y)\] 
   \[= (y + (x + x)).(x + (y + y))\] 
   \[= (y + 1).(x + 1)\] 
   \[= 1.1\] 
   \[= 1\] 
   universal bound law \(a + 1 = 1\)

   (b) Write the dual of the original statement.
(c) Write the original statement in the Boolean algebra of set theory.

5 points

5. Find, if possible, the truth values of the following statements:
   P = "Q is false"
   Q = "P is true"

5 points

6. There are 67 consecutively-numbered balls in an urn. The number on the last ball is 143. What is the number on the first ball?

5 points

7. A security system requires a password of 8 characters, which are either decimal digits, or upper-case letters.
   (a) How many such passwords are possible?

5 points
   (b) If we further require that the passwords use at least one digit, how many are now possible?

5 points
   (c) How many passwords have a '1' as the first character, or a '2' as the second character?

10 points

8. Which is more likely: To get a Head in one toss of a fair coin, or to get at least one 6 in 3 tosses of a fair die? Explain your answer carefully.

5 points

9. There are 10 students in an Academic Challenge group.
   (a) How many possible rankings of the top 4 students are there?

5 points
(b) How many different possible teams of 4 students could we select (leaving 6 students as alternates)?

5 points

(c) If Amber and Steve cannot both be on the team, how many teams are possible?

10 points

Final Examination

1. For any positive integers \(m, n\), consider the function \(f(m, n) = \gcd(m, n)\) (the greatest common divisor) function.

(a) Write the domain of this function, using appropriate set notation.

5 points

(b) What is an appropriate codomain for this function?

5 points

2. Define \(f : \mathbb{R} \rightarrow \mathbb{Z}\) by the piecewise rule \(f(x) = \begin{cases} 3 \left\lceil \frac{x}{2} \right\rceil + 5 & \text{if } x < 7 \\ \lfloor 2x \rfloor & \text{if } 7 \leq x \end{cases}\)

(a) Evaluate \(f(6.3)\)

5 points

(b) Evaluate \(f(8.999)\)

5 points

(c) What does this say about \(f\)?

5 points

3. Let \(A = \{a, b, c, d\}, B = \{1, 2, 3, 4, 5, 6\}\).

Explain your answers to the following:

(a) How many different functions are there from \(A\) to \(B\)?

10 points
(b) How many different one-to-one functions are possible from \(A\) to \(B\)?

10 points

(c) How many different onto functions are possible from \(A\) to \(B\)?

10 points

4. Carefully explain the answer to the following:

How many cards must you deal from a standard 52-card deck to guarantee that you have at least one 4 of a kind (i.e. 4 Aces or 4 Kings, etc)?

10 points

5. Define functions \(f, g : \mathbb{R} \to \mathbb{R}\) by the rules \(f(x) = \lfloor x + 0.5 \rfloor, g(x) = \lceil x - 0.5 \rceil\).

(a) Find and simplify \(f \circ g\).

5 points

(b) Find and simplify \(g \circ f\).

5 points

(c) Prove that \((f \circ g)(n) = (g \circ f)(n)\) for any integer \(n\).

10 points

(d) Find a counterexample to the claim that \(f \circ g = g \circ f\).

10 points

6. Define the sequence \(\{b_n\}\) by the rule \(b_1 = b_2 = b_3 = 1\) and \(b_n = n b_{n-1} - 2 b_{n-3}\) for \(n \geq 4\).

Evaluate \(b_6\).

10 points

7. Suppose that two people play the following game: Given a pile of \(n\) stones, they take turns removing either 1 or 3 stones. The player to take the last stone wins.

Write a recurrence relation to find \(g_n\), the total number of ways that the game could be played with \(n\) stones.
For example,
\( g_1 = 1 \) (first player takes 1 stone)
\( g_2 = 1 \) (first player takes 1 stone, other player takes last one)
\( g_3 = 2 \) (either the first player takes 3, or they take 1,1,1).

8. Define the sequence \( \{c_n\} \) by \( c_1 = 3 \), \( c_2 = 5 \), and \( c_n = 2c_{n-1} - c_{n-2} \) for \( n \geq 3 \).

By finding the first few terms of this sequence, find an explicit formula for \( c_n \).

9. Define a sequence recursively, via \( a_1 = 1 \) and \( a_{n+1} = 2a_n + 3 \) for \( n \geq 1 \).

Prove that \( a_n = 2^{n+1} - 3 \), using the Principle of Mathematical Induction.

10. Suppose that \( n \) is an integer, and that \( n \mod 7 = 3 \).

Evaluate \( (10n) \mod 7 \), and explain your answer.

11. Recall that the set difference can be expressed as \( A - B = A \cap B^c \).

Using this, translate the set identity \( A - B = A - (A \cap B) \) into the language of a boolean algebra.

12. Suppose that we have a universal statement of the form \((\forall x \in D)(P(x) \land Q(x) \rightarrow R(x))\).

(a) If we were to prove this by the contrapositive, what would we do?

(b) If we were to prove it instead by contradiction, what should we do?
13. Let $A$ be the truth set for the predicate $P(x)$, and $B$ be the truth set for the predicate $Q(x)$. If $A$ and $B$ are disjoint, which of the following are true, and why?

$P(x) \Rightarrow Q(x)$

$P(x) \Rightarrow \neg Q(x)$

$\neg P(x) \Rightarrow Q(x)$

$\neg P(x) \Rightarrow \neg Q(x)$. 

20 points