1. Given that \( y = 2x - 1 \) is a solution of \( (x^2 - x)y'' + (2x - 1)y' - 2y = 0 \), find a second linearly independent solution by reduction of order and write the general solution.

**Solution:** We have \( y_1 = 2x - 1, \ y_1' = 2, \ y_1'' = 0 \), so that
\[
(x^2 - x)y_1'' + (2x - 1)y_1' - 2y_1 = (2x - 1)(2) - 2(2x - 1) = 0
\]
Let \( y_2 = (2x - 1)v \), so that \( y_2' = 2v + (2x - 1)v' \) and \( y_2'' = 4v' + (2x - 1)v''. \) Feeding this solution into the differential equation yields
\[
(x^2 - x)(2x - 1)v'' + ((x^2 - x)4 + (2x - 1)2)v' = 0,
\]
or
\[
v'' + \left( \frac{4}{2x-1} + \frac{2x-1}{x^2-x} \right)v' = 0.
\]
The integrating factor is \( e^{\int \left( \frac{4}{2x-1} + \frac{2x-1}{x^2-x} \right) \ dx} = (2x - 1)^2(x^2 - x) \), from which
\[
\frac{d}{dx} \left( (2x - 1)^2(x^2 - x)v' \right) = 0.
\]
We integrate, and use partial fractions to obtain
\[
v' = \frac{A}{(2x-1)^2(x^2-x)} = A \left[ -\frac{4}{(2x-1)^2} - \frac{1}{x} + \frac{1}{x-1} \right]
\]
Integrating again yields
\[
v = A \left[ \frac{2}{2x-1} - \ln x + \ln(x - 1) \right] = A \left[ \frac{2}{2x-1} + \ln \left( \frac{x-1}{x} \right) \right]
\]
and so \( y_2 = (2x - 1)v = A \left[ 2 + (2x - 1) \ln \left( \frac{x-1}{x} \right) \right], \) and the general solution is
\[
y = A \left[ 2 + (2x - 1) \ln \left( \frac{x-1}{x} \right) \right] + B(2x - 1)
\]

2. Given that \( y = 3x + 5 \) is a solution to \( \left( \frac{9}{2}x^2 + 15x + 13 \right)y'' - 3(3x + 5)y' + 9y = 0 \), find a second linearly independent solution by reduction of order and write the general solution.

**Solution:** We have \( y_1 = 3x + 5, \ y_1' = 3, \ y_1'' = 0 \), so that
\[
\left( \frac{9}{2}x^2 + 15x + 13 \right)y_1'' - 3(3x + 5)y_1' + 9y_1 = -3(3x + 5)(3) + 9(3x + 5) = 0
\]
Let \( y_2 = (3x + 5)v \), so that \( y_2' = 3v + (3x + 5)v' \) and \( y_2'' = 6v' + (3x + 5)v''. \) Feeding this solution into the original differential equation yields
\[
\left( \frac{9}{2}x^2 + 15x + 13 \right)(3x + 5)v'' + \left( 6 \left( \frac{9}{2}x^2 + 15x + 13 \right) - 3(3x + 5)^2 \right)v' = 0,
\]
or
\[ v'' + \left( \frac{6}{3x+5} - \frac{3(3x+5)}{9\frac{x^2}{2} + 15x + 13} \right) v' = 0 \]

The integrating factor is \( \exp \left( 2 \ln(3x+5) - \ln \left( \frac{9}{2}\frac{x^2}{2} + 15x + 13 \right) \right) = \frac{(3x+5)^2}{\frac{9}{2}x^2 + 15x + 13} \).

from which
\[ \frac{d}{dx} \left( \frac{(3x+5)^2v'}{\frac{9}{2}x^2 + 15x + 13} \right) = 0 \]

We integrate, and divide, to obtain
\[ v' = A \left( \frac{\frac{9}{2}x^2 + 15x + 13}{(3x+5)^2} \right) = A \left( \frac{1}{2} + \frac{1}{2} \frac{1}{(3x+5)^2} \right) \]

We integrate again, and get
\[ v = A \left( \frac{x}{2} - \frac{1}{3x+5} \right), \]

from which \( y_2 = (3x+5)v = A \left( \frac{x}{2} (3x+5) - \frac{1}{6} \right) \) and the general solution is
\[ y = A \left( \frac{x}{2} (3x+5) - \frac{1}{6} \right) + B(3x+5). \]

3. Find the general solution to \( y^{(4)} + 19y^{(3)} + 126y'' + 324y' + 216y = 0 \)

Solution: This equation is linear, homogeneous, constant coefficient, with auxiliary equation \( m^4 + 19m^3 + 126m^2 + 324m + 216 = 0 \), which has zeros \(-1,-6,-6,-6\).

Hence, the general solution is \( y_c = c_1 e^{-x} + c_2 e^{-6x} + c_3 x e^{-6x} + c_4 x^2 e^{-6x}. \)

4. Solve the initial value problem \( y'' + 4y' + 68y = 0 : y(0) = 6, \quad y'(0) = -84. \)

Solution: The equation is linear, homogeneous, constant coefficient, with auxiliary equation \( m^2 + 4m + 68 = 0 \), which has zeros \(-2 \pm 8i\).

Hence, the general solution is \( y = c_1 e^{-2x} \cos(8x) + c_2 e^{-2x} \sin(8x). \)

From this we obtain \( y' = (8c_2 - 2c_1) e^{-2x} \cos(8x) - (8c_1 + 2c_2) e^{-2x} \sin(8x). \)

Using the given initial conditions, \( c_1 = 6, c_2 = -9, \) and the final solution is \( y = e^{-2x} (6 \cos(8x) - 9 \sin(8x)). \)