1. Find and sketch the domain of $f(x, y) = \frac{\sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)}{\sqrt{x + y}}$.

Solution: We require $x^2 + y^2 \geq 1$, $x^2 + y^2 < 4$ and $x + y > 0$. The region is graphed below. Solid lines are part of the domain, dashed ones are not.

2. Find the limit, or show that it does not exist: $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 3y^2}$

Solution: Using the fact that $\sin y \approx y$ as $y \to 0$, we write

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 3y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + 3y^2}$$

Now use polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, so the limit becomes

$$\lim_{r \to 0^+} \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^2 (\cos^2 \theta + 3 \sin^2 \theta)} = 0,$$

since $\lim_{r \to 0^+} r^2 = 0$ and $\frac{\cos^2 \theta \sin^2 \theta}{\cos^2 \theta + 3 \sin^2 \theta}$ is bounded for all $\theta$.

3. Show that $u = \sin(x - at) + \ln(x + at)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$.

Solution:

$$u_t = \cos(x - at) \cdot \frac{\partial}{\partial t}(x - at) + \frac{1}{x + at} \cdot \frac{\partial}{\partial t}(x + at) = -a \cos(x - at) + \frac{a}{x + at}$$

Differentiating again with respect to $t$: ...
\[u_{tt} = -a^2 \sin(x - at) - \frac{a^2}{(x + at)^2} = -a^2 \left( \sin(x - at) + \frac{1}{(x + at)^2} \right)\]

Similarly, \[u_{xx} = -\sin(x - at) - \frac{1}{(x + at)^2},\] so that \[u_{tt} = a^2 u_{xx},\] as desired.

4. Find all second partial derivatives of \(f(x, y) = \int_0^t e^{st} \, ds\).

**Solution:** Let \(G(t) = \int_0^t e^{st} \, ds\).

By the derivative form of the Fundamental Theorem of Calculus, \(G'(t) = e^{t^2}\).

By the integral form of the Fundamental Theorem of Calculus, we can write
\[f(x, y) = G(x + y) - G(xy)\].

Then, using the Chain rule:
\[\frac{\partial f}{\partial x} = G'(x + y) \cdot 1 - G'(xy) \cdot y = e^{(x+y)^2} - ye^{(xy)^2} = e^{(x+y)^2} - ye^{x^2y^2},\]
and
\[\frac{\partial f}{\partial y} = G'(x + y) \cdot 1 - G'(xy) \cdot x = e^{(x+y)^2} - xe^{x^2y^2}\].

Hence,
\[\frac{\partial^2 f}{\partial x^2} = 2(x + y)e^{(x+y)^2} - 2xy^3e^{x^2y^2}\],
\[\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = 2(x + y)e^{(x+y)^2} - (1 + 2x^2y^2)e^{x^2y^2}\],
\[\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = 2(x + y)e^{(x+y)^2} - (1 + 2x^2y^2)e^{x^2y^2}\],
and
\[\frac{\partial^2 f}{\partial y^2} = 2(x + y)e^{(x+y)^2} - 2x^3ye^{x^2y^2}\].