Rules:

• Show your work.
• Give exact answers.
• Cite any references used, other than your notes.
• You may not discuss this test with anyone but me.
• Hints cost one point each.

1. (5 points) Describe the image of the branch of the hyperbola \( x^2 - y^2 = 1, \ x \geq 1 \) under the mapping \( f(z) = e^{z^2} \).

2. (10 points) Let \( z \neq 0 \) be a fixed complex number. Show that the values of \( z^c \) all lie on a circle about the origin if and only if \( \text{Im } c = 0 \).

3. Let \( C \) be the boundary of the ellipse \( \frac{(x - 1)^2}{9} + \frac{(y - 1)^2}{4} = 1 \), traversed once with positive orientation. For any \( w \notin C \), define
   \[
   g(w) = \oint_{C} \frac{z^2 e^{3z}}{(z - w)^2} \, dz.
   \]
   (a) (5 points) Write \( g(2 - i) \) in the form \( a + bi \), and explain your answer.
   (b) (5 points) Write \( g(3) \) in the form \( a + bi \), and explain your answer.

4. Consider \( f(z) = \sin^6 z \).
   (a) (5 points) Use Euler’s formula for complex \( z \) to expand \( f(z) \) in terms of \( \cos 2z, \cos 4z \cdots \).
   (b) (5 points) Write the Maclaurin series (Taylor series with \( z = 0 \)) for \( f(z) \), in the form \( \sum_{n=0}^{\infty} c_n z^n \), where \( c_n \) is given explicitly.

5. (10 points) Suppose that \( f(z) \) is entire, and \( |f(z)| \leq |z e^{3z}| \) for all \( z \). Given that \( f(1) = 3 \), evaluate \( f'(2) \), and justify your answer.

6. (10 points) Let \( u(x, y) \) be harmonic on the disk \( D_1(1) \), and suppose that \( u \neq 3 \) for \( |z - 1| \leq 1 \). Is it necessarily true that both the maximum and minimum of \( |u - 3| \) for \( |z - 1| \leq 1 \) are achieved on \( C_1(1) \)? Explain your answer.

7. Let \( f(z) = \frac{1}{(z-1)(z-3)} \).
   (a) (5 points) Write the domains of every possible Laurent series centred at \( z = i \).
(b) (5 points) Use partial fractions to find the Laurent series for \( f(z) \) centred at \( z = i \) which is valid in the annulus containing \( z = 2 \).

8. Let 
\[ f(z) = \sum_{n=0}^{\infty} \left[ \frac{1}{(2n+1)!} + 6^n \right] \frac{1}{z^{2n}}. \]

(a) (5 points) Write a simplified formula for \( f(z) \) in terms of well-known functions.
(b) (5 points) Determine the region in which this Laurent series converges.

9. (5 points) Write a formula for a function \( f(z) \) which is analytic everywhere, except for a simple pole at \( z = i \), a pole of order 4 at \( z = 2 \), and an essential singularity at \( z = -1 \).

10. Calculate the following residues:
(a) (5 points) \( \frac{z^2 + 3}{z^3 + 4z^5 - 5} \) at \( z = 1 \).
(b) (5 points) \( \frac{\sin(z + 2)}{z^4} \) at \( z = 0 \).
   **Hint:** Use trigonometric identities.
(c) (5 points) \( z^{11} \exp \left( \frac{1}{z^2} \right) \) at \( z = 0 \)
(d) (5 points Graduate students/Bonus) \( \exp \left( z + \frac{1}{z} \right) \) at \( z = 0 \).

11. (5 points) Express 
\[ \oint_{C^+_2(0)} \frac{\exp(z^2)}{1 + 16z^4} \, dz \]
   in terms of residues.

**Note:** You do NOT need to calculate the residues.