Instructions:

1. Please include any references that you use.
2. Graduate students should do all problems, undergraduates should attempt 8 of the 10.

Problems

1. An *Euler circuit* of a graph $G$ is a circuit which visits each vertex, and uses each edge exactly once. 
   \textbf{Prove} that the complete graph $K_8$ does not have an Euler circuit.

2. A *Hamilton circuit* of a graph is a circuit which visits each vertex (except the terminal vertex) exactly once.
   Count the number of Hamilton circuits in the complete graph $K_n$ (for $n \geq 4$) which start at vertex $v_1$ and do not use the edge $\{v_1, v_3\}$.

3. Let $G_1$ and $G_2$ be graphs, and generate a new graph $G'$ by connecting exactly one vertex in $G_1$ to a vertex in $G_2$.
   Find a relation between the chromatic number $\chi(G')$ and $\chi(G_1), \chi(G_2)$.

4. Let $G$ be a graph, and $\omega$ be the size of the largest clique in $G$. Construct a new graph $G'$ from $G$ by adding one vertex, which is attached to every vertex in the clique.
   Find a relation between $P(x, G)$ and $P(x, G')$.

5. Suppose that $P(x, G)$ is the chromatic polynomial for a graph $G$, which has chromatic number $\chi(G) = k$.
   \textbf{Show} that $P(k, G)$ is a multiple of $k!$.

6. Given a positive integer $n$, we may construct a tree as follows:
   If $n = 1$ or $n$ is prime, we just create a single vertex with $n$.
   Otherwise, we find a factorization $n = pq$, where $1 < p \leq q < n$ (integers), and create a rooted tree with $n$ as the root, $p$ as the \textit{left child}, and $q$ as the \textit{right child}.
   We then repeat the process on the \textit{left subtree} and \textit{right subtree}.
   Count how many such trees there are for $n = 1, 2^1, 2^2, 2^3$, and try to deduce the number for $n = 2^4$.

7. Consider the adjacency matrix $A$ of the digraph $D_n$, which consists of $n$ vertices in a single cycle.
   \textit{Recall} that the entries of $A^k$ count paths.
   What can you say about $A^n$?
8. Write the ordinary generating function whose coefficients are the number of ways of making \( n \$ \) in change, using pennies, nickels, dimes, quarters and half-dollars.

9. Suppose that we have a group of 100 voters before an election. Of the 100, 46 intend to vote for George Bush, 44 for Al Gore and 10 for Pat Buchanan.

Find the ordinary generating function which counts the number of possible outcomes if only \( n \) of the 100 voters make it to the polls.

You may treat all voters in a group as indistinguishable.

10. Suppose that a certain certification requires 12 hours of training, with at least 3 in a simulator, at least 2 in a classroom, and at least 1 in the field.

If hours spent in one area are equivalent, use an exponential generating function to find the number of ways that the training could be completed.