1. For any positive integers $m, n$, consider the function $f(m, n) = \gcd(m, n)$ (the greatest common divisor) function.

(a) Write the domain of this function, using appropriate set notation.

(b) What is an appropriate codomain for this function?

2. Define $f : \mathbb{R} \to \mathbb{Z}$ by the piecewise rule $f(x) = \begin{cases} 
3 \lceil \frac{x}{2} \rceil + 5 & \text{if } x < 7 \\
2x & \text{if } 7 \leq x
\end{cases}$

(a) Evaluate $f(6.3)$

(b) Evaluate $f(8.999)$

(c) What does this say about $f$?

3. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5, 6\}$.

Explain your answers to the following:

(a) How many different functions are there from $A$ to $B$?
(b) How many different one-to-one functions are possible from \( A \) to \( B \)?

(c) How many different onto functions are possible from \( A \) to \( B \)?

4. Carefully explain the answer to the following:

How many cards must you deal from a standard 52-card deck to guarantee that you have at least one 4 of a kind (i.e. 4 Aces or 4 Kings, etc)?

5. Define functions \( f, g : \mathbb{R} \to \mathbb{R} \) by the rules \( f(x) = \lfloor x + 0.5 \rfloor \), \( g(x) = \lceil x - 0.5 \rceil \).

(a) Find and simplify \( f \circ g \).

(b) Find and simplify \( g \circ f \).

(c) Prove that \( (f \circ g)(n) = (g \circ f)(n) \) for any integer \( n \).

(d) Find a counterexample to the claim that \( f \circ g = g \circ f \).
6. Define the sequence \( \{b_n\} \) by the rule \( b_1 = b_2 = b_3 = 1 \) and \( b_n = n b_{n-1} - 2 b_{n-3} \) for \( n \geq 4 \). Evaluate \( b_6 \).

7. Suppose that two people play the following game: Given a pile of \( n \) stones, they take turns removing either 1 or 3 stones. The player to take the last stone wins.

Write a recurrence relation to find \( g_n \), the total number of ways that the game could be played with \( n \) stones.
For example,
\[
\begin{align*}
g_1 &= 1 \text{ (first player takes 1 stone)} \\
g_2 &= 1 \text{ (first player takes 1 stone, other player takes last one)} \\
g_3 &= 2 \text{ (either the first player takes 3, or they take 1,1,1).}
\end{align*}
\]

8. Define the sequence \( \{c_n\} \) by \( c_1 = 3 \), \( c_2 = 5 \), and \( c_n = 2c_{n-1} - c_{n-2} \) for \( n \geq 3 \).
By finding the first few terms of this sequence, find an explicit formula for \( c_n \).
9. Define a sequence recursively, via \( a_1 = 1 \) and \( a_{n+1} = 2a_n + 3 \) for \( n \geq 1 \).

Prove that \( a_n = 2^{n+1} - 3 \), using the Principle of Mathematical Induction.

10. Suppose that \( n \) is an integer, and that \( n \mod 7 = 3 \).

Evaluate \( (10^n) \mod 7 \), and explain your answer.

11. Recall that the set difference can be expressed as \( A - B = A \cap B^c \).

Using this, translate the set identity \( A - B = A - (A \cap B) \) into the language of a boolean algebra.

12. Suppose that we have a universal statement of the form \((\forall x \in D)(P(x) \land Q(x) \rightarrow R(x))\).

(a) If we were to prove this by the contrapositive, what would we do?

(b) If we were to prove it instead by contradiction, what should we do?
13. Let $A$ be the truth set for the predicate $P(x)$, and $B$ be the truth set for the predicate $Q(x)$. If $A$ and $B$ are disjoint, which of the following are true, and why?

$P(x) \Rightarrow Q(x)$

$P(x) \Rightarrow \neg Q(x)$

$\neg P(x) \Rightarrow Q(x)$

$\neg P(x) \Rightarrow \neg Q(x)$.