1. (16 points) Find the general solution to \( \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = xe^{-2x} + 8x \)

Solution: The auxiliary equation is \( m^2 + 4m + 4 = (m+2)^2 \), with zeros \( m = -2, -2 \), so that \( y_c = c_1 e^{-2x} + c_2 xe^{-2x} \).

We try \( y_p = Ax^3 e^{-2x} + Bx^2 e^{-2x} + Cx + D \)
\( y'_p = 3Ax^2 e^{-2x} - 2Ax^3 e^{-2x} + 2Bxe^{-2x} - 2Bxe^{-2x} + C \)
\( y''_p = 6Ax e^{-2x} - 12Ax^2 e^{-2x} + 4Ax^3 e^{-2x} + 2B e^{-2x} - 8B xe^{-2x} + 4B x^2 e^{-2x} \)
Then, \( y''_p + 4y'_p + 4y_p = 6Ax e^{-2x} + 2B e^{-2x} + 4C x + 4C + 4D = xe^{-2x} + 8x \),
from which \( A = \frac{1}{6}, B = 0, C = 2, D = -2 \), and hence
\( y = c_1 e^{-2x} + c_2 xe^{-2x} + \frac{1}{6} x^3 e^{-2x} + 2x - 2 \)

2. (16 points) Given that \( y = \frac{\cos(3x)}{\sqrt{x}} \) is a solution of \( x^2 y'' + xy' + \left( 9x^2 - \frac{1}{4} \right) y = 0 \), find the general solution of this differential equation.

Solution: We apply reduction of order, with \( y_1 = \frac{\cos(3x)}{\sqrt{x}} \) and \( P(x) = \frac{x}{x^2} = \frac{1}{x} \)

Hence, \( y_2 = \frac{\cos(3x)}{\sqrt{x}} \int \frac{x}{\cos^2(3x)} \exp \left( - \int \frac{1}{x} \, dx \right) \, dx = \frac{\cos(3x)}{\sqrt{x}} \int \sec^2(3x) \, dx \)
\( = \frac{\cos(3x)}{3\sqrt{x}} \tan(3x) = \frac{\sin(3x)}{3\sqrt{x}} \)
Hence, then general solution is \( y = c_1 \frac{\cos(3x)}{\sqrt{x}} + c_2 \frac{\sin(3x)}{3\sqrt{x}} \)

3. (16 points) Find the general solution to \( x^4 y'' - 6x^2 y = 1 - 6x^2 \).

Solution: We first look at the homogeneous equation, to find \( y_c \), namely \( x^4 y'' - 6x^2 y = 0 \). Dividing by \( x^2 \) yields the Cauchy-Euler equation \( x^2 y'' - 6y = 0 \).
Setting \( y = x^m \) yields the auxiliary equation \( m(m-1) - 6 = (m-3)(m+2) = 0 \).
Thus, \( y_c = c_1 x^3 + c_2 x^{-2} \).
We apply variation of parameters, with \( f(x) = x^{-4} - 6x^{-2}, y_1 = x^3, y_2 = x^{-2} \).
The Wronskian is \[ W = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -5. \]

Then,
\[
y_p = -x^3 \int \frac{x^{-2}(x^{-4} - 6x^{-2})}{-5} \, dx + x^{-2} \int \frac{x^3(x^{-4} - 6x^{-2})}{-5} \, dx
\]
\[
= \frac{x^3}{5} \left( -\frac{x^{-5}}{5} + 6x^{-3} \right) - \frac{x^{-2}}{5} \left( \ln|x| - 3x^2 \right) = 1 - \frac{1}{5}x^{-2} \ln|x| - \frac{x^{-2}}{25}.
\]

Notice that the last term is part of \( y_c \), so we write the general solution as
\[
y = c_1 x^3 + c_2 x^{-2} + 1 - \frac{1}{5}x^{-2} \ln|x|.
\]

4. (16 points) A solution to \( y'' + 6y' + 58y = 130e^x \) is \( y = 2e^x \).

A solution to \( y'' + 6y' + 58y = 65e^x + 10 + 60x + 290x^2 \) is \( y = e^x + 5x^2 \).

Find the general solution to \( y'' + 6y' + 58y = 1 + 12x + 58x^2 \).

Solution: The auxiliary equation is \( m^2 + 6m + 58 = 0 \), with zeros \( m = -3 \pm 7i \).

Thus, \( y_c = c_1 e^{-3x} \cos(7x) + c_2 e^{-3x} \sin(7x) \).

Since \( 1 + 12x + 58x^2 = \frac{1}{5}(65e^x + 10 + 60x + 290x^2) - \frac{1}{10}(130e^x) \), we have \( y_p = \frac{1}{5}(e^x + 5x^2) - \frac{1}{10}(2e^x) = x^2 \).

Thus, the general solution is \( y = c_1 e^{-3x} \cos(7x) + c_2 e^{-3x} \sin(7x) + x^2 \).

5. (16 points) Solve the system of equations \( \frac{dx}{dt} = 12x - 17y \), \( \frac{dy}{dt} = 4x - 4y \).

Solution: \( |A - \lambda I| = \begin{vmatrix} 12 - \lambda & -17 \\ 4 & -4 - \lambda \end{vmatrix} = \lambda^2 - 8\lambda + 20 = 0 \), with eigenvalues \( \lambda = 4 \pm 2i \).

Solving \( \begin{bmatrix} 12 - (4 + 2i) \\ 4 \end{bmatrix} \begin{bmatrix} -17 \\ -4 - (4 + 2i) \end{bmatrix} K = 0 \) yields the eigenvector \( K = \begin{bmatrix} 17 \\ 8 - 2i \end{bmatrix} = \begin{bmatrix} 17 \\ 8 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix} \).

Thus,
\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{4t} \begin{Bmatrix} 17 \\ 8 \end{Bmatrix} \cos(2t) - c_2 e^{4t} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \sin(2t)
\]
\[
+ c_2 e^{4t} \begin{Bmatrix} 17 \\ 8 \end{Bmatrix} \sin(2t) + c_2 e^{4t} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \cos(2t)
\]

6. A 32 lb weight stretches a spring 32 ft. This weight is then lifted 1 ft above equilibrium and released. There is a damping in this system that is equal to twice the instantaneous velocity. Further, a force equal to \( 2(\cos t + \sin t) \) lbs acts on the weight.
(a) (8 points) Write down the governing differential equation and initial conditions for the motion of the weight. DO NOT SOLVE THE EQUATION.

Solution: The mass is \( m = \frac{32}{32} = 1 \) slug, and \( k = \frac{32}{32} \) lb/ft, so the equation is
\[
1x'' + 2x' + x = 2(\cos(t) + \sin(t)), \quad x(0) = -1, \quad x'(0) = 0.
\]

(b) (4 points) The answer to the above equation is \( x(t) = c_1e^{-t} + c_2te^{-t} + \sin t - \cos t \). Find \( c_1 \) and \( c_2 \).

Solution: \( x' = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} + \cos(t) + \sin(t) \).
This, \( c_1 - 1 = -1, c_1 = 0 \) and \( c_2 + 1 = 0, c_2 = -1 \)

(c) (4 points) SET UP BUT DO NOT SOLVE the mathematical expression you would use to determine when the transient part of the solution is less than 0.1% of the amplitude of the steady-state part of the solution.

Solution: Find \( t \) so that \( te^{-t} \leq 0.001 \sqrt{2} \)

(d) (4 points) SET UP BUT DO NOT SOLVE the mathematical expression you would use to determine the times when the mass changes its direction of motion.

Solution: Find \( t \) so that \( x'(t) = -e^{-t} + te^{-t} + \cos(t) + \sin(t) = 0 \)
1. (16 points) Find the general solution to $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 4\cos x + 12x$.

2. (16 points) A solution to $4xy'' + 2y' + y = -3\sin \sqrt{x}$ is $y = \frac{3}{2}\sqrt{x}\cos \sqrt{x}$. A solution to $4xy'' + 2y' + y = 0$ is $y = \cos \sqrt{x}$. Find the general solution to $4xy'' + 2y' + y = \sin \sqrt{x}$.

3. (16 points) Solve the system of equations
\[
\begin{align*}
\frac{dx}{dt} &= 2x - 4y \\
\frac{dy}{dt} &= -x - y.
\end{align*}
\]

4. (16 points) Find the general solution to $y'' + 3y' + 2y = \frac{1}{1 + e^x}$.

5. (16 points) Find the solution to $x^2y'' - 3xy' + 4y = 0$, subject to $y(1) = 2$ and $y'(1) = 3$.

6. A 64 lb weight is attached to a spring hanging from the ceiling. This causes the spring to stretch 0.519 ft on coming to rest at equilibrium. There is damping numerically equal to 1/5 the instantaneous velocity in this system.

Initially the weight is released 3 ft above the equilibrium position with an upward velocity of 40.64 ft/sec.

(a) (6 points) Write down the governing differential equation and initial conditions for the motion of the weight. DO NOT SOLVE THE EQUATION.

(b) (6 points) The answer to the above differential equation is $x(t) = e^{-t/20} \left[-3\cos(7.85t) - 3\sqrt{3}\sin(7.85t)\right]$.

Find all the times when the mass passes through the equilibrium position and indicate which times correspond to upward motion and which to downward motion.

(c) (8 points) Find the time $T$, to within 1 second, such that $|x(t)| < 0.1$ ft for $t > T$. 