1. (a) Find a counterexample to the following statement:
If $A$, $B$ and $C$ are sets satisfying $A \subset B$ and $A \subset C$, then $B \subseteq C$.

(b) Prove that, for any sets $A$ and $B$:
$A = A \cap (A \cup B)$

2. For each $n \in \mathbb{N}$, define $A_n = \mathbb{N} - \{3n, 3n + 1, 3n + 2, \ldots, 4n\}$.

(a) Write a listing of the elements in $A_1, A_2, A_3$.

(b) Find $\bigcup_{n=1}^{\infty} A_n$

(c) Find $\bigcap_{n=1}^{\infty} A_n$
3. For each $n \in \mathbb{N}$, define $B_n = \left( -\frac{1}{n}, 3 - \frac{1}{n} \right)$.

We prove that $\cap_{n=1}^{\infty} B_n = [0, 2)$ by the following steps:

(a) Prove that $[0, 2) \subseteq \cap_{n=1}^{\infty} B_n$.

(b) Prove that $x \geq 2 \Rightarrow x \notin \cap_{n=1}^{\infty} B_n$

(c) Prove that $x < 0 \Rightarrow x \notin \cap_{n=1}^{\infty} B_n$

4. (a) State the PMI (Principle of Mathematical Induction), the PCI (Principle of Complete Induction), and the WOP (Well Ordering Principle).
(b) Use the PMI to prove that \(2^n \geq n + 1\) for \(n \geq 1\).

(c) Consider the recurrence relation \(a_1 = 3, a_2 = 8, a_n = 2a_{n-1} - a_{n-2}\) for \(n \geq 3\).
Use the PCI to prove that \(a_n = 5n - 2\) for \(n \geq 1\).

(d) Grade the following proof. If the grade is not 'A', give your reasons.

Claim: If \(a\) and \(b\) are natural numbers satisfying \(a \leq b\), then \(na \leq nb\) for all \(n \in \mathbb{N}\).

"Proof". Suppose not.
Then, we can find natural numbers \(a\), \(b\) and \(n\) satisfying \(a \leq b\) and \(na > nb\).
Then, \(T\) is a non-empty subset of \(\mathbb{N}\), which must have a smallest element, say \(m\), by the WOP.
\(m \neq 1\), since this would imply \(1a = a > 1b = b\).
Thus, \(m - 1 \geq 1\) and \(m - 1 \notin T\).
Hence, \((m - 1)a \leq (m - 1)b\) and \(ma \leq mb\).
But, \(ma = (m - 1)a + m \leq (m - 1)b + a \leq (m - 1)b + b = mb\), which is a contradiction.
Thus, the set \(T\) must be empty •
5. (a) A mathematics department has 20 full-time tenure-track faculty members, 8 full-time instructors, and a pool of 30 part-time faculty. Of the instructors, 5 are also in the part-time pool.

How many faculty members are there to cover classes?

(b) In how many ways could we rank 8 of the 25 students in a class?

(c) From a group of 100 education students, we wish to select a team of 6. If there are 65 women in the group, how many such teams are there with exactly 4 women?

(d) Suppose that we wish to form a committee of 5 people from a department of 30 people. Professor Smith and Professor Jones cannot work together on a committee.

How many committees can we form?

(e) Use the Binomial Theorem to evaluate \[ \sum_{k=1}^{37} \binom{37}{k} 4^k \]