1. (a) Claim: For sets $A, B, C$,

$$A \cup (B - C) = (A \cup B) - (A \cup C)$$

Find a counterexample to this claim.

(b) Prove the following: If $A, B$ are sets, and $A \subseteq B$ then $A \cup B = B$. 

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2. For each positive integer $n$, let $A_n = [0, \infty) - (n, 3n]$.

(a) Write $A_1, A_2, A_3$ as a union of intervals.

(b) Find and simplify $\bigcup_{n=1}^{\infty} A_n$.

(c) Find and simplify $\bigcap_{n=1}^{\infty} A_n$. 

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3. Let \( S = \{aaaaaa, aaaaab, \ldots, zzzzzz\} \) be the set of 6-letter 'words'.

Give exact expressions for the following:

(a) The number of elements in \( S \).

(b) The number of elements of \( S \) whose letters are all different.

(c) The number of elements of \( S \) whose first letter or last letter are vowels (a,e,i,o,u).

(d) The number of elements of \( S \) whose letters are all different, and in alphabetic order, e.g. actuvz

(e) The number of elements of \( S \) that have at least two a’s.
4. (a) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ for all positive integers $n$.

(b) Suppose that $a_1 = 17$, $a_2 = 77$ and $a_{n+2} = 6a_{n+1} - 5a_n$ for $n \geq 1$.
Prove that $a_n = 2 + 3(5^n)$ for all positive integers $n$. 

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5. Explain or correct the steps in the following proof:

**Archimedean Principle.** For all positive integers \( a, b \), there exists a positive integer \( n \) such that \( na \geq b \).

**Proof:**

Suppose not.

Then, there exist \( \hat{a}, \hat{b} \in \mathbb{N} \)

such that \( n\hat{a} < \hat{b} \) for all \( n \in \mathbb{N} \)

Let \( T = \{ \hat{b} - n\hat{a} : n \in \mathbb{N} \} \)

Then, \( T \subseteq \mathbb{N} \) and \( T \neq \emptyset \)  \hspace{1cm} \text{Why?}

Then, \( T \) has a smallest element, \( t \).  \hspace{1cm} \text{Why?}

There exists \( m \in \mathbb{N} \) such that \( t = \hat{b} - m\hat{a} \)  \hspace{1cm} \text{Why?}

But, \( \hat{b} - (m + 1)\hat{a} \in T \) and  \hspace{1cm} \text{Why?}

\( \hat{b} - (m + 1)\hat{a} = t - \hat{a} < t \)

This is a contradiction,  \hspace{1cm} \text{Why?}

so our hypothesis must be false.