When Does a Metric Generate Convex Balls?

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Abstract

Given a metric imposed on a real linear vector space, we explore the issue of convexity of the open balls generated by the metric.

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1 Introduction

The concept of a *metric* is usually first encountered in beginning courses in real analysis, presented as an abstraction to more general sets of the concept of distance in Euclidean space.

This abstraction allows for the discussion of such issues as convergence and continuity, and extends to the even more general ideas of point-set topology.

When one deals with real linear spaces (vector spaces over the reals), it is usual to discuss the concept of *norm*, which abstracts the notion of length, rather than distance.

However, there are cases in which real linear spaces may be endowed with a metric, rather than a norm. In such cases, it seems reasonable to ask how the algebraic structure of the linear space and metric combine to generate a geometry on the space.

One natural question which arises is the following :

What metric spaces have the property that all open balls in the space are convex?

To my chagrin, this question arose because of an error that I made in the final examination of a graduate–level analysis course, which I did not notice until I was constructing the answer key.

Our starting point is the question which I originally posed, and an example which demonstrates that not all metrics have the property that we seek.

We follow that with a complete characterization of the metrics with this property, and some special cases, which provide weaker sufficient conditions that are easier to verify.

We continue by discussing some other convexity notions which appear in the literature. Namely, the topological concept of *local convexity* (Willard [2], p. 164), and the functional concept of a *(metrically) convex space* (Bryant [1]), and show that both of these notions are independent of the problem under consideration.

Finally, we discuss two particular constructions. The first shows that there are metrics in which some of the open balls are convex, and some are not. The second construction is, in some sense, an inverse problem. Given any convex set which is symmetric with respect to the origin, we construct a number of metrics whose open balls are dilations, contractions or translations of this set.

2 Convex Open Balls in Metric Spaces

As discussed above, the question addressed here appeared on an examination that I gave in analysis, and led me to the subsequent investigation.

The question posed was to prove the following :

Proposition 2.1 *If E is a linear space and ρ is a metric on E, then the open ball $B(\mathbf{x}; r) = \{\mathbf{y} \in E : \rho(\mathbf{x}, \mathbf{y}) < r\}$ is convex.*

To see that this is indeed false in general, consider the following :

Proposition 2.2 *Consider the linear space $E = \mathbb{R}^2$, with the metric*

$$\rho(\mathbf{x}, \mathbf{y}) = \rho((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|} .$$

Then, $B(\mathbf{x}; r)$ is not convex for any $\mathbf{x} \in E$, $r > 0$.

Figure 1: The unit ball $B(\mathbf{0};1) = \{\mathbf{y} \in \mathbb{R}^2 \ : \ \sqrt{|y_1|} + \sqrt{|y_2|} < 1\}$

Proof. Clearly, ρ is a metric on E, which can be proven using the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, for $a, b \geq 0$.

Given any $\mathbf{x} \in E$ and $r > 0$, choose $\frac{r^2}{2} < \alpha < r^2$, and let $\mathbf{y} = (x_1 + \alpha, x_2)$, $\mathbf{z} = (x_1, x_2 + \alpha)$.
Then, $\rho(\mathbf{x}, \mathbf{y}) = \sqrt{\alpha} = \rho(\mathbf{x}, \mathbf{z}) < r$, so that $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}; r)$.
However, $\rho\left(\mathbf{x}, \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}\right) = \sqrt{2\alpha} > r$, so that $\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z} \notin B(\mathbf{x}; r)$. *Q.E.D.*

In Figure 1, we present the unit ball $B(\mathbf{0};1)$ in this metric, which is clearly not a convex set.

Having established that I did indeed make a mistake in this question, it is natural to explore necessary and sufficient conditions for the validity of Proposition 2.1, and related problems.

At this point, it is appropriate to note again that, while the questions which arise are quite natural, and the results obtained in this paper are elementary, an exhausting search of the literature has failed to find this specific issue addressed.

3 Necessary and Sufficient Conditions

Given that our previous example shows that not all metrics generate convex open balls, we are in a position to completely characterize those metrics which do so.

Theorem 3.1 *Proposition 2.1 holds if and only if*

$$\rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) \leq \max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{x}, \mathbf{z})\} \ ,$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ *and* $0 \leq t \leq 1$.

Proof. On one hand, suppose that the indicated condition holds.
That is, given any $\mathbf{x} \in E$, with $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}; r)$ and $0 \leq t \leq 1$, we have

$$\rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) \leq \max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{x}, \mathbf{z})\} < r \ ,$$

so that $t\mathbf{y} + (1-t)\mathbf{z} \in B(\mathbf{x}; r)$, the desired convexity condition.
On the other hand, if there exist $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ and $0 \leq t \leq 1$ such that

$$\rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) > \max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{x}, \mathbf{z})\} \ ,$$

we may choose

$$\max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{x}, \mathbf{z})\} < r < \rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) \ ,$$

from which $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}; r)$, but $t\mathbf{y} + (1-t)\mathbf{z} \notin B(\mathbf{x}; r)$, meaning that the latter set is not convex. *Q.E.D.*

Note that, if ρ satisfies this condition, $\mathbf{x} \in E$, and P is any convex polytope in E, then the function $f \ : \ P \to [0, \infty)$ defined by

$$f(\mathbf{y}) = \rho(\mathbf{x}, \mathbf{y})$$

attains its maximum value at one of the vertices of P.
However, this optimization condition does not categorize the metrics that we seek, since it can be satisfied by functions which are not metrics, such as $f(\mathbf{x}, \mathbf{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2$ on $\mathbb{R}^2$.
While Theorem 3.1 gives a complete description of the metrics which generate convex balls, the functional condition given is, in practice, difficult to verify.
Consequently, it is appropriate at this point to consider some special cases.

Corollary 3.2 *If any of the following conditions holds on a metric ρ on a linear space E, then Proposition 2.1 holds. That is, the open balls of the metric are convex.*

1. *ρ is the* discrete metric*. That is, $\rho(\mathbf{x}, \mathbf{y}) = 1$ if $\mathbf{x} \neq \mathbf{y}$.*
2. *For each fixed $\mathbf{x} \in E$, ρ is a convex function on E.*

   That is, for all $\mathbf{y}, \mathbf{z} \in E$ and $0 \leq t \leq 1$:

$$\rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) \leq t\rho(\mathbf{x}, \mathbf{y}) + (1-t)\rho(\mathbf{x}, \mathbf{z})$$

3. *ρ satisfies the following conditions :*
   - *(a)* $\rho(\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{z}) = \rho(\mathbf{y}, \mathbf{z})$ *for all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$.
     (ρ is translation invariant*).*
     and
   - *(b)* $\rho(t\mathbf{x}, t\mathbf{y}) \leq t\rho(\mathbf{x}, \mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in E$ *and* $0 \leq t \leq 1$.
4. *There exists a norm $||.||$ on E such that $\rho(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ for $\mathbf{x}, \mathbf{y} \in E$.*

Proof.

1. Suppose that ρ is the discrete metric. Then,
   $B(\mathbf{x}; r) = \{\mathbf{x}\}$ for any $\mathbf{x} \in E$ and $0 \leq r \leq 1$
   and
   $B(\mathbf{x}; r) = E$ for any $\mathbf{x} \in E$ and $1 < r$.
   In either case, $B(\mathbf{x}; r)$ is clearly convex.
2. Suppose that the indicated condition holds.
   Then, for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ and $0 \leq t \leq 1$, we have

   $\rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) \leq t\rho(\mathbf{x}, \mathbf{y}) + (1-t)\rho(\mathbf{x}, \mathbf{z}) \leq \max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{x}, \mathbf{z})\}$,

   which, by Theorem 3.1 means that all open balls are convex.

3. Suppose that the given pair of conditions holds.

   We will show that this implies the conditions of 2.

   For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ and $0 \leq t \leq 1$, we have

$$\begin{aligned} \rho(\mathbf{x}, t\mathbf{y} + (1-t)\mathbf{z}) &= \rho(\mathbf{0}, t(\mathbf{y}-\mathbf{x}) + (1-t)(\mathbf{z}-\mathbf{x})) \\ &\leq \rho(\mathbf{0}, t(\mathbf{y}-\mathbf{x})) + \rho(t(\mathbf{y}-\mathbf{x}), t(\mathbf{y}-\mathbf{x}) + (1-t)(\mathbf{z}-\mathbf{x})) \\ &\leq t\rho(\mathbf{0}, \mathbf{y}-\mathbf{x}) + \rho(\mathbf{0}, (1-t)(\mathbf{z}-\mathbf{x})) \\ &\leq t\rho(\mathbf{x}, \mathbf{y}) + (1-t)\rho(\mathbf{0}, \mathbf{z}-\mathbf{x}) \\ &= t\rho(\mathbf{x}, \mathbf{y}) + (1-t)\rho(\mathbf{x}, \mathbf{z}) \ , \end{aligned}$$

   the desired result.

4. Suppose that ρ is derived from a norm. Then, it clearly satisfies the conditions of 3, since it is translation invariant, and also satisfies the stronger condition $\rho(t\mathbf{x}, t\mathbf{y}) = t\rho(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in E$ and $0 \leq t$. *Q.E.D.*

4 Other Convexity Concepts

While there are at least two other concepts involving convexity which appear in the literature, we show in this section that these concepts are independent of the central question of this work.
While there is a class of metric spaces called *(metrically) convex* spaces, the open balls in such spaces are not necessarily convex, since the definition (given below) is clearly independent of the geometry of the space.

Definition 4.1 *(cf. Bryant [1]) A metric space* (X, ρ) *is* (metrically) convex *if and only if, given distinct points* $\mathbf{x}, \mathbf{y} \in X$*, there exists* $\mathbf{z} \in X$*, different from both* $\mathbf{x}$ *and* $\mathbf{y}$*, such that*

$$\rho(\mathbf{x}, \mathbf{z}) + \rho(\mathbf{z}, \mathbf{y}) = \rho(\mathbf{x}, \mathbf{y}) \ .$$

On one hand, by considering again the discrete metric, it is clear that a metric on a linear space which generates convex open balls is not necessarily convex.

On the other hand, the metric of Proposition 2.2 satisfies the Intermediate Value Theorem, and so is a convex metric, but generates no open convex balls.

We may thus deduce that the definition of convex metric is independent of the convexity of the open balls of the metric.

One can also consider the topology that is induced by the metric, which leads one to the concept of a *locally convex* space (Willard [2], p.164), in which each point of the space has a base of convex sets.

This standard definition leads immediately to the following result :

Proposition 4.2 *If ρ is a metric on the linear space E such that all open balls are convex, then the induced linear topological space is locally convex.*

On the other hand, if the metric ρ induces a locally convex topology on the linear space E, it is not necessarily true that all open balls are convex. In fact, it can be the case that *no* open balls in a locally convex metric space are themselves convex.

To see this, consider the following :

Example 4.3 *Let $E = \mathbb{R}^2$, and ρ be the metric of Proposition 2.2. Then, none of the open balls is convex, but the metric space is locally convex.*

In Proposition 2.2, it was shown that no open balls are convex in this case. However, it is clear that each open ball contains the interior of a disk (a convex set) and that each such disk can be expressed as a union of non-convex open balls. Hence, each such disk is an open set, and forms a base for the metric topology, which means that the topology is locally convex.

Just as in the above, this demonstrates that local convexity is independent of the convexity of the open balls of the metric.

5 Two Constructions

To conclude, we provide two constructions. The first demonstrates that there exist metrics on linear spaces which have some open balls which are convex, and some which are not. The second shows that, given a convex set in a real linear space which is symmetric with respect to the origin, it is possible to construct a metric whose open balls are dilations, contractions or translations of this convex set.

We recall the following result, which appears in many introductory texts, frequently as an exercise :

Theorem 5.1 *If ρ_1 and ρ_2 are metrics on a set E, then the following is also a metric on E :*

$$\rho'(\mathbf{x},\mathbf{y}) = \max\{\rho_1(\mathbf{x},\mathbf{y}), \rho_2(\mathbf{x},\mathbf{y})\} \ .$$

Using this result, it is possible to construct metric spaces in which some of the open balls are convex, and others are not. To illustrate, consider :

Example 5.2 *Let $E = \mathbb{R}^2$, ρ_1 be the metric of Proposition 2.2, and ρ_2 be the Euclidean metric. That is,*

$\rho_1(\mathbf{x},\mathbf{y}) = \sqrt{|x_1 - y_1|}+\sqrt{|x_2 - y_2|}$ *and* $\rho_2(\mathbf{x},\mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
Then, using the metric $\rho' = \max\{\rho_1, \rho_2\}$, the corresponding open ball $B'(\mathbf{x}; r)$ is not convex for $0 < r < \sqrt{8}$, and is convex if $\sqrt{8} \leq r$.

To see this, it suffices to make two observations :

If $B_1(\mathbf{x}; r)$, $B_2(\mathbf{x}; r)$ and $B'(\mathbf{x}; r)$ denote the open balls centred at $\mathbf{x}$ with radius r in the metrics ρ_1, ρ_2 and ρ' respectively, then

$$B'(\mathbf{x}; r) = B_1(\mathbf{x}; r) \cap B_2(\mathbf{x}; r) \tag{5.1}$$

and, via some simple analysis,

$$B_1(\mathbf{x}; r/8^{1/4}) \subset B_2(\mathbf{x}; r^2/8^{1/2}) \subset B_1(\mathbf{x}; r) \tag{5.2}$$

for any $0 < r$. Moreover, the inclusions in (5.2) are sharp.

Now, if $\sqrt{8} \leq r$, then, by (5.1) and (5.2), we have $B'(\mathbf{x}; r) = B_2(\mathbf{x}; r)$, which is clearly convex, since it is the interior of a disk. On the other hand, if $0 < r \leq 1$, then $B'(\mathbf{x}; r) = B_1(\mathbf{x}; r)$, which is not convex, as demonstrated in Proposition 2.2. For $1 < r < \sqrt{8}$, $B'(\mathbf{x}; r)$ is a region whose boundary has 8 arcs, 4 of which are portions of the boundary of $B_1(\mathbf{x}, r)$, and hence cannot be convex, by our previous observations.

Figure 2: The open ball $B'(\mathbf{0}; 5/2)$

To clarify, Figure 2 shows the ball $B'(\mathbf{0}; 5/2)$.

As claimed above, we conclude by constructing a metric from a particular convex set.

In a general space, quantities such as *boundary*, *boundedness*, *closed set* and the *interior* of a set require that a metric be imposed a priori. However, in the case of a linear space, we may implicitly define these quantities by means of the underlying algebraic structure.

Let E be a real linear space, and C be a subset of E which satisfies the following conditions :

i) C is convex.

ii) For each $\mathbf{x} \in E$, $\mathbf{x} \neq \mathbf{0}$, there exists a unique $T = T(\mathbf{x}) > 0$ such that

$$L(\mathbf{x}) \ := \ \{t \in \mathbb{R} \ : \ \mathrm{t}\mathbf{x} \in \mathrm{C}\} \ = \ [-T, T] \ .$$

That is, each line through the origin intersects C in a line segment which is symmetric about $\mathbf{0}$. Examples of such sets in $\mathbb{R}^{\mathrm{n}}$ include closed spheres and closed polytopes symmetric with respect to $\mathbf{0}$.

Define the set B in E via

$$B \ = \ \{T(\mathbf{x})\mathbf{x} \ : \ \mathbf{x} \neq \mathbf{0}\} \ .$$

B is the *boundary* of C, in the topology that we generate, and the set $C - B = \{\mathbf{x} \in C \ : \ \mathbf{x} \notin B\}$ its interior.

Note that, for any $t \geq 0$, the sets $tC = \{t\mathbf{x} \ : \ \mathbf{x} \in C\}$ and $t(C - B)$ are also convex sets.

Furthermore, let $f \ : \ (0, \infty) \rightarrow (0, \infty)$ be a function which satisfies the following conditions :

i) $f(t)$ is non-decreasing.

ii) For all $t_1, t_2 \geq 0$, $f(t_1 + t_2) \leq f(t_1) + f(t_2)$.

Examples of such functions include $f(t) = t^p$ for $0 \leq p \leq 1$, and piecewise-defined functions such as

$$f(t) \ = \ \begin{cases} \frac{1}{3} & if \quad 0 < t < 1 \\ 1 & if \quad 1 \leq t \end{cases}$$

Using the properties of the set B, and the functions T and f, we may define a metric ρ on E via

$$\rho(\mathbf{x}, \mathbf{y}) \ = \ \begin{cases} 0 & if \quad \mathbf{x} = \mathbf{y} \\ f\left(1/T(\mathbf{x} - \mathbf{y})\right) & if \quad \mathbf{x} \neq \mathbf{y} \end{cases}$$

To verify that this is a metric, we check the conditions of the definition :

1. $\rho(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in E$, $\rho(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

   This follows from the definition of ρ, using the facts that T and f are strictly positive functions.

2. $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in E$.

   This follows from the symmetry condition on C, which implies that $T(-\mathbf{x}) = T(\mathbf{x})$.

3. $\rho(\mathbf{x}, \mathbf{z}) \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$.

   This condition clearly holds if any two of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are equal. If not, we set $a = 1/T(\mathbf{x} - \mathbf{y})$ and $b = 1/T(\mathbf{y} - \mathbf{z})$, from which $\rho(\mathbf{x}, \mathbf{y}) = f(a)$,

$\rho(\mathbf{y},\mathbf{z}) = f(b)$, and there exist $\mathbf{u},\mathbf{v} \in B$ such that $\mathbf{x}-\mathbf{y} = a\mathbf{u}$ and $\mathbf{y}-\mathbf{z} = b\mathbf{v}$. Hence,

$$\begin{aligned} \mathbf{x}-\mathbf{z} &= (\mathbf{x}-\mathbf{y}) + (\mathbf{y}-\mathbf{z}) \\ &= a\mathbf{u} + b\mathbf{v} \\ &= (a+b)\left(\tfrac{a}{a+b}\mathbf{u} + \tfrac{b}{a+b}\mathbf{v}\right) \end{aligned}$$

By the convexity of C, it follows that the vector $\mathbf{w} = \dfrac{a}{a+b}\mathbf{u} + \dfrac{b}{a+b}\mathbf{v} \in C$, whence $T(\mathbf{w}) \geq 1$, and so $1/T(\mathbf{w}) \leq 1$.

Thus, $\mathbf{x}-\mathbf{z} = (a+b)\mathbf{w} \in (a+b)C$, from which

$$\rho(\mathbf{x},\mathbf{z}) \leq f(a+b) \leq f(a) + f(b) = \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z}) \ ,$$

the desired inequality.

To illustrate, consider :

Example 5.3 *Let E be the real linear space $\mathbb{R}^2$, C be the closed hexagon with vertices $\{(1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1)\}$ and $f(t) = \sqrt{t}$. Then, the set B is the boundary of this polygon. Given the point $\mathbf{x} = (2, 0.5)$, we have $0.4\mathbf{x} \in B$, so that $T(\mathbf{x}) = 0.4$ and hence $\rho(\mathbf{0},\mathbf{x}) = \sqrt{2.5}$. This is illustrated in figure 3.*

Having thus defined the metric ρ, we may make the following observations :

i) By the definition, ρ is translation invariant.

ii) With the choice $f(t) = t$, ρ generates a norm on E.

iii) For any $r > 0$ and $\mathbf{x} \in E$, the open ball $B(\mathbf{x}, f(r))$ is precisely the set $r(C - B)$, which, as noted above, is convex.

References

[1] V. W. Bryant. The convexity of the subset space of a metric space. *Compositio Math.*, 22 : 383–385, 1970.

[2] S. Willard. *General Topology.* Addison-Wesley Publishing, 1970.

Figure 3: The convex polygon and point $(2, 0.5)$