Lagrange Multipliers - Extremizing the volume of a box

Problem: Given a closed rectangular box, of surface area $S$ and total edge length $L$, extremize the volume of the box.

Let the lengths of the edges be $x, y$ and $z$. Then, the problem is to extremize $xyz$ subject to the conditions

$$2xy + 2yz + 2zx = S$$
$$4x + 4y + 4z = L$$

We set up the Lagrangian

$$F(x, y, z, \lambda, \mu) = xyz - \lambda(2xy + 2yz + 2zx - S) - \mu(2x + 2y + 2z - L)$$

which gives us the set of equations

$$yz - \lambda(2y + 2z) - 4\mu = 0$$
$$xz - \lambda(2x + 2z) - 4\mu = 0$$
$$xy - \lambda(2x + 2y) - 4\mu = 0$$
$$2xy + 2yz + 2zx = S$$
$$4x + 4y + 4z = L$$

Subtracting the first 3 equations pairwise and factoring yields

$$(y - x)(z - 2\lambda) = 0$$
$$(z - y)(x - 2\lambda) = 0$$
$$(x - z)(y - 2\lambda) = 0$$

Then $x = y$ or $z = 2\lambda$. In the second case, the other two equations then become

$$(2\lambda - y)(x - 2\lambda) = 0$$
$$(x - 2\lambda)(y - 2\lambda) = 0$$

which gives $x = z$ or $y = z$, and is equivalent to $x = y$.

We thus may assume that $x = y$, which gives the two cases:

Case 1: $x = y = z$ (a cube). This is only possible if $L^2 = 24S$, and gives only one possible solution $x = y = z = \frac{L}{12}$; $V = xyz = \frac{L^3}{1728}$.

Case 2: $x = y = 2\lambda$. Solving for $z$ in the equation for the edge length yields $z = \frac{L - 16\lambda}{4}$.

We substitute these values into the equation for the surface area and obtain after simplification

$$24\lambda^2 - 2L\lambda + S = 0$$

Examining the discriminant of this quadratic shows that we only have a solution if

$$L^2 - 24S \geq 0$$
Putting all of these back into the original equations yields 2 distinct solutions

\[
D = \sqrt{L^2 - 24S}
\]
\[
x = \frac{1}{12}(L \pm D)
\]
\[
y = \frac{1}{12}(L \pm D)
\]
\[
z = \frac{1}{12}(L \mp 2D)
\]
\[
\lambda = \frac{1}{24}(L \pm D)
\]
\[
\mu = -\lambda^2 = -\frac{1}{576} (2L^2 - 24S \pm 2LD)
\]

The extreme volumes are then given by

\[
V_{\text{min}} = \frac{1}{1728} (24LS - 2D^3)
\]
\[
V_{\text{max}} = \frac{1}{1728} (24LS + 2D^3)
\]

In the special case when \(S = 1500\ cm^3\), \(L = 200\ cm\), we obtain (to 3 decimal places)

\[
x = 21.937\ cm
\]
\[
y = 21.937\ cm
\]
\[
z = 6.126\ cm
\]
\[
\lambda = 10.969
\]
\[
\mu = -120.309
\]
\[
V_{\text{min}} = 2947.937\ cm^3
\]

and

\[
x = 11.396\ cm
\]
\[
y = 11.396\ cm
\]
\[
z = 27.208\ cm
\]
\[
\lambda = 5.698
\]
\[
\mu = -32.468
\]
\[
V_{\text{max}} = 3533.544\ cm^3
\]