

Homework Set 5

Due date: Thursday 3 April

Type your responses to the extent possible. If necessary, leave blank space in the document to write equations by hand. **Your answers to the interpretation questions should be brief – you don't need to write long paragraphs.**

Zombie Apocalypse. In the beginning, there were only Humans. But then, on March 1, 2014 (a day known ever after simply as $t = 0$), it happened. A solar flare from the sun caused a random genetic mutation in a small fraction of the Human population, and turned them into ... (loud blaring music) ... Latents. These unfortunate individuals looked and acted Human, but were infected with zombiism, and steadily, inexorably, inevitably turned into ... (louder blaring music) ... Zombies. It is entirely likely that you yourself will become a Zombie – ok, well, it depends on which Alternate Reality, listed below, you end up in. But still, it could happen. After the initial solar flare, the only way for a Human to become a Latent is through a Human-Zombie interaction (a process best left to your imagination, dear reader).

Denote the Human population by $H(t)$, scaled by millions, so the initial Human population of 7.1 billion is represented as $H(0) = 7100$ megapeople. Denote the Latent population by $L(t)$, also scaled by millions. The initial Latent population of 1 million people is $L(0) = 1$ megaperson. Denote the Zombie population by $Z(t)$, also scaled by millions. Time t is measured in years. In the problems below, you will formulate and analyze a variety of HLZ life cycles (ok, death cycles) to determine whether the Human race will survive or whether the Zombies will rule ... at least until the next asteroid strike.

0. (10 point bonus). Let's start with an easy one. Come up with an awesome name for the Zombie Overlord.
1. (20 pts) Alternate Reality #1: Latents Live Among Us.

In this Reality, Latents cannot be identified before the onset of zombiism. This means that they can participate in the natural reproduction process.

- a. Build the 3 compartment model for the population system based on the following assumptions:
 - Humans and Latents reproduce exponentially (ie, at a rate proportional to the Human plus Latent population with no logistic term) with rate constant (birth rate) a . The standard birth rate for the entire homework is $a = 0.02$ (2% growth per year). All babies are born as Humans. Ignore natural Human death.
 - Humans become Latents through a standard-form interaction between Humans and Zombies, with rate constant (putrification rate) b . The standard putrification rate for the entire homework is $b = 10^{-4}$.
 - Latents are converted into Zombies at a steady rate proportional to the Latent population, with rate constant (zombification rate) c . The standard zombification rate for the entire homework is $c = 0.2$. Zombies cannot be destroyed.

- b. What is the maximum value that c can have? Explain.
 - c. Does the system have any critical points? What does this mean in terms of the 3 populations?
 - d. Will the human race ever die out? Explain qualitatively in words and/or using inequalities. Provide just a brief explanation.
2. (10 pts) Alternate Reality #2: Somehow, We Know.

In this Reality, Latents cannot be explicitly identified before the onset of zombiism, but subconsciously Humans can tell; the impact is that Latents are effectively barred from the reproduction process. Modify the model of Reality #1 to account for this change. This model behaves similarly to the first, so there's no point in repeating the questions.

3. (20 pts) Alternate Reality #3: Die, Zombie, Die.

In this Reality, Zombies can be destroyed, at a rate proportional to the Zombie population, with rate constant (death rate) d . The standard death rate for the entire homework is $d = 0.3$. As in Reality #2, Latents are not able to reproduce.

- a. Modify the model of Reality #2 by including the zombie death term.
 - b. Find the equilibrium condition that must be satisfied for all 3 populations to remain constant. Using the standard numerical values of the rate constants listed above, find the number of Humans, Latents and Zombies that lead to equilibrium. These won't sum to 7100 – this is an Alternate Reality.
 - c. If the populations are not balanced as in part b, will the human race ever die out? Explain qualitatively in words and/or using inequalities. Provide just a brief explanation.
4. (30 pts) Alternate Reality #4: Latent-Free Zone.

In this Reality, there are no Latents. The initial solar flare generates $Z(0) = 1$ megazombie with the initial Human population still at $H(0) = 7100$. Humans transform directly in Zombies when infected. Zombies can be destroyed.

- a. Modify the model of Reality #3 by removing the Latents and using the standard Human-Zombie interaction term to convert Humans to Zombies. This is a 2×2 system.
- b. Find the critical points of the system in terms of the symbols a , b and d . Then use the standard values for a , b and d to determine numerical values for the equilibrium populations of Humans and Zombies.
- c. Perform linear stability analysis on each of the critical points to classify them (center, stable or unstable node, stable or unstable spiral, or saddle).
- d. To help you visualize the direction field without computer graphics, show what happens for these two special trajectories: (1) initially, there are $Z(0) = Z_0 > 0$ megazombies but no Humans, and (2) initially, there are $H(0) = H_0 > 0$ megapeople but no zombies. Draw a rough sketch of the phase plane using the results of (c) and these 2 trajectories.

- e. Draw a rough sketch of the phase plane trajectory for the initial condition $H(0) = 7100$, $Z(0) = 1$.
 - f. Suppose that Humans create a more effective way of destroying Zombies, so that now $d = 0.6$. How will the equilibrium populations be affected?
5. (30 pts) Alternate Reality #5: The Multiple Zombie Premise.
- In this modification of Reality #4, Zombies are weaker, so that it takes 2 Zombies working in concert to convert a Human into a Zombie.
- a. Modify the model of Reality #4 by changing the interaction term.
 - b. Find the critical points of the system in terms of the symbols a , b and d . Then use the standard values for a , b and d to determine numerical values for the equilibrium populations of Humans and Zombies.
 - c. Perform linear stability analysis on the nonzero critical point to classify it (center, stable or unstable node, stable or unstable spiral, or saddle). Use the symbolic form here, and simplify the matrix before finding the eigenvalues.
 - d. Draw a rough sketch of the phase plane, and draw the special trajectories where (1) $Z(0) = 0$ and $H(0) = H_0 > 0$ and (2) $H(0) = 0$ and $Z(0) = Z_0 > 0$. Then pick an initial condition close to the nonzero critical point, and sketch the trajectory on the same plot.
 - e. What will happen to the Human and Zombie populations in the long term?