3450:436/536 Mathematical Models, Kreider, Spring 2014

Homework Set 3

Due date: Monday 24 February

Type your responses to the extent possible. If necessary, leave blank space in the document to write equations by hand.

1. (20 pts) Consider the equation $x^3-(4+\epsilon)x+2\epsilon = 0$. Find the first order corrections to the solutions $x_B = 0$ and $x_C = 2$ with $\epsilon = .001$. Explain the odd thing that happens with x_C.

2. (20 pts) For the initial value problem $y' = -y + \epsilon y^2$, $y(0) = 1$, we derived the asymptotic solution $y(\tau) = e^{-\tau} + \epsilon\left(e^{-\tau} - e^{-2\tau}\right) + \epsilon^2\left(e^{-\tau} - 2e^{-2\tau} + e^{-3\tau}\right)$. Because this is a Bernoulli equation, it is possible to find the exact solution, $y(\tau) = \dfrac{e^{-\tau}}{1+\epsilon(e^{-\tau}-1)}$. Use the geometric series $\dfrac{1}{1+A} = 1 - A + \dfrac{1}{2}A^2 + \cdots$ to show that the leading terms in the exact solution match the asymptotic solution up to second order. This provides some measure of validation to the perturbation approach.

3. (20 pts) Find the first order perturbation solution to $y'(\tau) = -\epsilon y + y^2$, $y(0) = 1$.

4. (20 pts) Consider the harvesting problem $\dfrac{dP}{dt} = aP - bP^2 - H$, $P(0) = P_0$, which we nondimensionalized to $\dot{\tilde{P}} = \tilde{P} - \epsilon\tilde{P}^2 - \beta$, with $\epsilon = P_0/M$ and $\beta = H/aP_0$. Suppose that the harvesting term is small, $O(\epsilon)$, so that $\beta = \alpha\epsilon$. (a) Write both the $O(1)$ and $O(\epsilon)$ perturbation equations. (b) Use the $O(1)$ solution to demonstrate that for very early time the population grows when β is $O(\epsilon)$. (c) Find the $O(\epsilon)$ solution. What feature in that solution suggests that the population might start to decline as time goes by? (d) Using the parameter values presented in class, $a = 0.5$, $b = 10^{-5}$, $P_0 = 1000$, determine how many fish per year, H, can be harvested when $\beta = \alpha\epsilon$, assuming that α is 1. Note that this is a short time limitation; as the population grows, more harvesting can safely be done.

5. 536 STUDENTS ONLY. (20 pts) Find the first order perturbation solution to $y''(t) + 2y'(t) + \epsilon y(t) = 0$ with $y(0) = 0$, $y'(0) = 1$. For the first order equation, recall that in the method of undetermined coefficients, if there is duplication between the homogeneous solutions and the standard form of the particular solution, the particular solution must be multiplied by the lowest power of t that avoids the duplication.