3450:436/536 Mathematical Models, Kreider, Spring 2014

Homework Set 2 – UPDATED VERSION

Due date: Wednesday 12 February

Type your responses to the extent possible. If necessary, leave blank space in the document to write equations by hand.

1. (30 pts) In class, we derived the equation $mx'' + \beta x' - ks = 0$ for a linear spring, in which the spring's restoring force is $F = -k(s + x)$. Another type of spring, called a 'hard spring', has the property that it stiffens more with extension. The restoring force is generally assumed to have the form $F = -k(s + x) - bx^3$. Consider initial conditions $x(0) = A$ and $x'(0) = 0$.

   a) Write the dimensional form of the ODE for a hard spring.

   b) Identify the units on b.

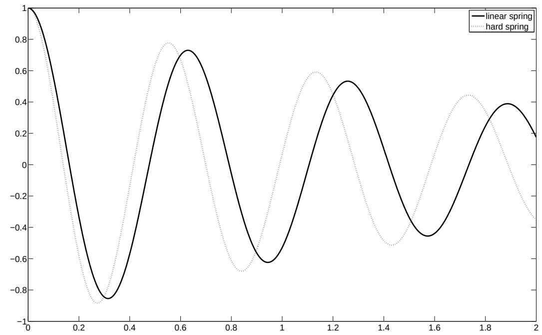
   c) Scale x by $x(0) = A$, and scale t by an unknown T. Nondimensionalize the equation so that $\ddot{\tilde{x}}$ has coefficient 1.

   d) Find the time scale T that normalizes the nonlinear term, to make sure that the nonlinearity remains in the problem. Substitute this into the nondimensional equation and show that each term in the equation is dimensionless.

   e) It can be difficult to identify when a term in the equation is small enough to neglect, because the magnitude of the derivatives is not always apparent. A rough rule of thumb is that a term can be neglected if its magnitude is 10% of the others. To estimate the size of the derivatives, let's try the following. Drop the damping and nonlinear terms, so only the inertial (second derivative) and linear (with k) terms remain (yes, it's a bit questionable). The solution to that equation is $\cos\left(\frac{\sqrt{k/b}}{A}\tau\right)$ (with the initial conditions applied). This implies that $\tilde{x}$ can be as big as 1. Take the first and second derivatives to see how big $\dot{\tilde{x}}$ and $\ddot{\tilde{x}}$ can get. Go back to the full nondimensional equation and use these values to identify the largest magnitude that each of the 4 terms can have, written in terms of the symbols A, m, k, β, and/or b. Now compare the inertial (second derivative) and damping (first derivative) terms. Find the expression for β that makes the damping term equal 10% of the inertial term.

   f) 536 STUDENTS ONLY (extra 5 points). Use the line of reasoning in (e) to compare the linear and nonlinear spring terms (k and b). Identify the condition for which the linear term is at least 10 times bigger than the nonlinear term.

2. (20 pts) Write a formal paragraph explaining why a hard spring oscillates slightly more than a linear spring. In the figure below, the nonlinearity was exaggerated for effect: $A = 1$, $m = 1$, $k = 100$, $b = 50$, $\beta = 1$. As the spring is extended, the restoring force grows rapidly, causing the spring to accelerate back to equilibrium. But there is relatively little damping, so when x is small, there is no force to retard the motion, causing the spring to overshoot. The purpose of this problem is to give you experience in writing an argument. My intention is to give you some feedback and have you resubmit the paragraph to gain any points you missed the first time.



3. (20 pts) In class, we derived the projectile at altitude initial value problem

$$
\begin{aligned}
x''(t) &= \frac{-g}{(1+x/R)^2} \\
x(0) &= 0 \\
x'(0) &= V
\end{aligned}
$$

a) In our original scaling, $y = x/R$, $\tau = t/T$, $T = R/V$, $\epsilon = V^2/gR$, explain why there is a problem in the limit as $\epsilon \to 0$. What is the physical interpretation of ϵ being very small?

b) In our original scaling, explain what happens as $\epsilon \to \infty$. What has to happen to $\ddot{y}$ for the equation to make sense? What does that correspond to physically?

c) To fix the problem in part a, let's rescale the variables so that we can handle a small ϵ. First, scale x as $y = x/L$ where L is a characteristic length. Use the Buckingham theorem to find a combination of V and g that provides a length $L = f(V, g)$. Second, identify the time scale T so that $\tau = t/T$ nondimensionalizes the equation (remember, the goal is to simplify the coefficients in the equation as much as possible). Third, write the nondimensional equation (with its initial conditions) and identify the dimensionless grouping ϵ. Fourth, show what happens to the equation as $\epsilon \to 0$. Finally, solve the simplified nondimensional equation. What does this remind you of (think back to beginning calculus)?

4. (20 pts) Consider the final formula for the length of a day in the Hours of Daylight section: $H = \dfrac{\pi + E}{2\pi} \cdot 24$, with E given in lecture.

a) How do you identify the longest day of the year?

b) Find the latitude of Akron and use the formula to estimate the length of the longest day of the year.

c) Find a set of major cities in the northern hemisphere (pick your favorites) that lie at latitudes roughly $+10^o$, $+20^o$, $+30^o$, $+50^o$ and $+60^o$. Use the formula to estimate the length of the longest day of the year in those cities. There is a MATLAB code on my web site that you are welcome to use.