3450:730 Advanced Numerical Solution of Partial Differential Equations, Kreider, Summer 2008

Homework Problem 10

Due date: Tuesday 10 June 2008

9. (30 points) Consider this Burger's Equation problem:

$$\begin{aligned} u_t &= \alpha u_{xx} - u u_x \qquad 0 < x < 1, t > 0 & (1) \\ u(x,0) &= \sin(\pi x) & (2) \\ u(0,t) &= 0 & (3) \\ u(1,t) &= 0 & (4) \end{aligned}$$

Part A.

In preparation for Computing Project 2, discretize this nonlinear equation using Crank-Nicolson with successive approximations. First, solve the PDE for u_{xx} to get

$$\alpha u_{xx} = u_t + u u_x \tag{5}$$

On the left, average the central differences at time steps n and $n+1$. On the right, use central differences with half step on u_t, average u at time steps n and $n+1$, and average u_x using central differences at time steps n and $n+1$. To set up the successive approximation equations, keep the u_{xx} terms at $n+1$ on the left and move everything else to the right. Denote u_i^{n+1} as $v_i^{(m+1)}$ on the left and as $v_i^{(m)}$ on the right (do the same for $i \pm 1$ as well). This will be the primary equation for the upcoming code. The homework problem is to write the primary equation in a form suitable for developing code (unknowns on the left and known quantities on the right so you can build the matrix and right-hand side, ignore the boundary conditions for now). You will discover in the computing project that this formulation is unstable, so we will fix it in Part B below.

Part B.

To stabilize the formulation, bring the u_i^{n+1} from u_t over to the left. Write the appropriate form of the primary equation. You will discover that this small change (which makes the formulation a bit more 'implicit') is enough to make the successive approximations converge quite nicely. The homework problem is to write the primary equation in a form suitable for developing code.