100 Points

Diff Eq - 3450:335
EXAM #1 S06

NAME Key
ROW

Show **ALL** your work.

1. Solve $\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$.

Bernoulli

$y^{-3}\dfrac{dy}{dx} + \dfrac{2}{x}y^{-2} = x^{-2}$ 3

Let $z = y^{-2} \Rightarrow \dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$ 3

$-\dfrac{1}{2}\dfrac{dz}{dx} + \dfrac{2}{x}z = x^{-2}$

$\dfrac{dz}{dx} - \dfrac{4}{x}z = -2x^{-2}$ 2

IF: $e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = x^{-4}$ 2

$x^{-4}\dfrac{dz}{dx} - 4x^{-5}z = -2x^{-6}$

$\dfrac{d}{dx}\left[x^{-4}z\right] = -2x^{-6}$

Integrate 2

$x^{-4}z = \dfrac{2}{5}x^{-5} + C$

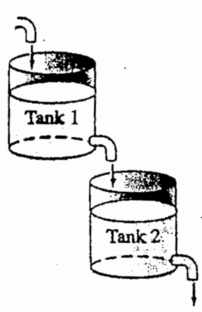
$z = \dfrac{2}{5}x^{-1} + Cx^{4}$ 2

So $y^{-2} = \dfrac{2}{5}x^{-1} + Cx^{4}$

1

15 Points

2a. Two tanks are designed as shown in the figure. Tank 1 initially contains 500 gallons of water with 70 kg of sugar in solution. Water containing 4 kg of sugar per gallon is entering the tank at the rate of 5 gal/min. The well-stirred mixture is flowing out of the tank at the rate of 9 gal/min. If A(t) denotes the amount of sugar in kg in the tank at any time t in minutes, SET UP BUT DO NOT SOLVE the differential equation(s) and condition(s) necessary to find A(t).



$A(0) = 70$ 1

$\dfrac{dA}{dt} = 4\,\dfrac{\text{kgs}}{\text{gal}}\cdot 5\,\dfrac{\text{gal}}{\text{m}} - \dfrac{9A}{500+(5-9)t}$

1 2 3

6 Points

2b. Tank 2 initially contains 500 gallons of pure water. The well-stirred mixture is flowing out of the tank at the rate of 6 gal/min. If B(t) denotes the amount of sugar in kg in the tank at any time t in minutes, SET UP BUT DO NOT SOLVE the differential equation(s) and condition(s) necessary to find B(t).

$B(0) = 0$ 1

$\dfrac{dB}{dt} = \dfrac{9A}{500-4t} - \dfrac{6B}{500+(9-6)t}$

1 2 2

6 Points

27 Points

3. Solve $\frac{dy}{dx} = (y - e^x)^2 - 3(y - e^x) + 2 + e^x$ by making the substitution $z(x) = y - e^x$. You do not need to solve for y in your final answer.

$z = y - e^x$

$\frac{dz}{dx} = \frac{dy}{dx} - e^x \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} + e^x$ 2

15 Points

Subst: LHS

$\frac{dz}{dx} + \cancel{e^x} = z^2 - 3z + 2 + \cancel{e^x}$ 4

$\frac{dz}{dx} = z^2 - 3z + 2 = (z-2)(z-1)$ separable

$\frac{dz}{(z-1)(z-2)} = dx$

$\left[\frac{-1}{z-1} + \frac{1}{z-2}\right]dz = dx$ 4

$-\ln|z-1| + \ln|z-2| = x + c$

$\ln\left|\frac{z-2}{z-1}\right| = x + c$ 4

$\ln\left|\frac{y-e^x-2}{y-e^x-1}\right| = x + c$

1

4. Solve $\left(\frac{r}{\theta^2} + \frac{1}{r^2} - \sin(2\theta)\right)\frac{d\theta}{dr} = \left(\frac{1}{\theta} + \frac{2\theta}{r^3}\right)$.

$\underbrace{\left(\frac{r}{\theta^2} + \frac{1}{r^2} - \sin 2\theta\right)}_{M}^{F_\theta} d\theta + \underbrace{-\left(\frac{1}{\theta} + \frac{2\theta}{r^3}\right)}_{N}^{F_r} dr = 0$

15 Points

$M_r = \frac{1}{\theta^2} - \frac{2}{r^3} = N_\theta = \frac{1}{\theta^2} - \frac{2}{r^3}$ so exact 5

4

$F_\theta = \frac{r}{\theta^2} + \frac{1}{r^2} - \sin 2\theta \Rightarrow F = -\frac{r}{\theta} + \frac{\theta}{r^2} + \frac{1}{2}\cos 2\theta + A(r)$

$F_r = -\frac{1}{\theta} - \frac{2\theta}{r^3} + A'(r) = -\frac{1}{\theta} - \frac{2\theta}{r^3}$

so $A'(r) = 0$ 4

$\Rightarrow A =$ constant

Hence $F(r,\theta) = -\frac{r}{\theta} + \frac{\theta}{r^2} + \frac{1}{2}\cos(2\theta) = C$

2

30 Points

5. Solve $\frac{dy}{dx} = \frac{ye^{x/y} + y\sin\left(\frac{x}{y}\right)}{y + xe^{x/y} + x\sin\left(\frac{x}{y}\right)}$. Homogeneous

$$\left(y + xe^{x/y} + x\sin\left(\tfrac{x}{y}\right)\right)dy = \left(ye^{x/y} + y\sin\tfrac{x}{y}\right)dx$$

let $x = yv \Rightarrow dx = y\,dv + v\,dy$ 4

15 Points

$$\left[y + \cancel{yve^{v}} + \cancel{yv\sin v}\right]dy = \left[ye^{v} + y\sin v\right](y\,dv + v\,dy)$$
$$= \left(y^2e^{v} + y^2\sin v\right)dv + \left(\cancel{yve^{v}} + \cancel{yv\sin v}\right)dy$$

$$y\,dy = y^2\left(e^{v} + \sin v\right)dv$$

$$\frac{1}{y}\,dy = \left(e^{v} + \sin v\right)dv \quad 5$$

$$\ln|y| + c = e^{v} - \cos v \quad 4$$

$$\ln|y| + c = e^{x/y} - \cos\left(\tfrac{x}{y}\right)$$

2

6. A hot air balloon is hovering 1000 meters above the ground. An object of mass 2 kg is dropped from the balloon. The air leads to a drag resistance $\left(\text{in units of } \frac{\text{kg·m}}{\text{s}^2}\right)$ that is equal to $\frac{4V(t)}{t+1}$, where V(t) is the velocity of the object at any time t.

3

Find the velocity of the object at any time t.

$$m\frac{dv}{dt} = mg - \frac{4v}{t+1} \qquad v(0) = 0 \quad 2$$

15 Points

$$2\frac{dv}{dt} + \frac{4v}{t+1} = 2g \quad \text{Linear}$$

$$\frac{dv}{dt} + \frac{2}{t+1}v = g$$

$$IF = e^{\int \frac{2}{t+1}dt} = e^{2\ln|t+1|} = (t+1)^2 \quad 2$$

$$(t+1)^2\frac{dv}{dt} + 2(t+1)v = g(t+1)^2$$

2

$$\frac{d}{dt}\left[(t+1)^2 v\right] = g(t+1)^2$$

Integrate

$$(t+1)^2 v = g\frac{(t+1)^3}{3} + c$$

$$v = \frac{g(t+1)}{3} + \frac{c}{(t+1)^2} \quad 4$$

$$V(0) = 0 \Rightarrow 0 = \frac{g}{3} + c \Rightarrow c = -\frac{g}{3}$$

$$\text{so } v(t) = \frac{g(t+1)}{3} - \frac{g}{3(t+1)^2} \quad 2$$

30 Points

7a. A differential equations student finds a can of soda on the kitchen counter. The temperature of the soda is 50°F when the student finds it. Ten minutes later the temperature of the soda is 60°F, and ten minutes after that the temperature is 65°F. SET UP BUT DO NOT SOLVE the differential equation(s) and condition(s) to determine the temperature, T(t), of the soda at any time t.

$T(0) = 50$

$T(10) = 60$

$T(20) = 65$

3

$\frac{dT}{dt} = k(T_{kitchen} - T)$

1

1

5 Points

7b. The solution of your differential equation is $T(t) = 70 - 20e^{-kt}$. Find k.

$T(10) = 60 = 70 - 20e^{-10k}$

2

$-10 = -20e^{-10k}$

$e^{10k} = 2$

$10k = \ln 2$

$\Rightarrow k = \frac{1}{10}\ln 2$

1

3 Points

7c. What is the temperature of the kitchen?

steady state $\Rightarrow t \to \infty$

so $T_{kitchen} = 70°F$

2Points

7d. The student reasons that the soda was originally in the refrigerator. The temperature of the refrigerator is 30°F. At what time was the soda taken out of the refrigerator?

$30 = 70 - 20e^{(-\frac{1}{10}\ln 2)t}$

2

$-40 = -20e^{(-\frac{1}{10}\ln 2)t}$

$2 = e^{(-\frac{1}{10}\ln 2)t}$

$\ln 2 = (-\frac{1}{10}\ln 2)t$

1

so $t = -10$ minutes before entering room

3 Points

13 Points