30 min

Diff Eq - 3450:335
FINAL Fall01

NAME Key
ROW

150 Points

Show **ALL** your work.

5 1/2

1. Find the general solution to $y^2\frac{dx}{dy} + 3xe^{1/y} = x^2e^{1/y}$.

15 Points

$$y^2\frac{dx}{dy} = (x^2-3x)e^{1/y}$$

$$\frac{dx}{x(x-3)} = \frac{dy\, e^{1/y}}{y^2} = dx\left[\frac{-1/3}{x} + \frac{1/3}{x-3}\right] \quad (4)$$

$$-e^{1/y} + C = -\frac{1}{3}\ln x + \frac{1}{3}\ln|x-3| \quad (8)$$

$$e^{\frac{1}{y}} = \frac{1}{3}\ln x - \frac{1}{3}\ln|x-3| + C$$

$$\frac{1}{y} = \ln[\qquad]$$

$$y = \frac{1}{\ln[\quad]} \quad (3)$$

2. Solve $(x^3+2)\frac{dy}{dx} - 12x^2y = 18x^5 + 36x^2$ subject to $y(0) = 28$.

15 Points

Linear $\frac{dy}{dx} - \frac{12x^2}{x^3+2}y = 18x^2\frac{\cancel{(x^3+2)}}{\cancel{x^3+2}}$ -5, +4

$$IF = e^{\int\frac{-12x^2}{x^3+2}dx} = e^{-4\ln|x^3+2|} = (x^3+2)^{-4} \quad (5)$$

$$\frac{d}{dx}\left[(x^3+2)^{-4}y\right] = 18x^2(x^3+2)^{-4} \quad (3)$$

$$(x^3+2)^{-4}y = -2(x^3+2)^{-3} + C \quad (5)$$

$$y = -2(x^3+2) + C(x^3+2)^4$$

$$y(0) = 28 = -4 + 16C$$

$$32 = 16C \text{ so } C = 2 \quad (2)$$

$$y = -2(x^3+2) + 2(x^3+2)^4$$

30 Points

3. Solve $\frac{dy}{dt} - 4y = 7e^{-3t} + \delta(t-2)$ subject to $y(0) = 5$.

$$sY - y(0) - 4Y = \frac{7}{s+3} + e^{-2s} \quad (5)$$

$$Y = \frac{5}{s-4} + \frac{7}{(s+3)(s-4)} + \frac{e^{-2s}}{s-4}$$

$$\frac{-1}{s+3} + \frac{1}{s-4} \quad (2)$$

$4\frac{1}{2}$

$$y(t) = 5e^{4t} - 1e^{-3t} + e^{4t} + U(t-2)e^{4(t-2)}$$

(1) (1) (1) (2)

12Points

4. Find the general solution to $xy'' + 7y' + \frac{25y}{x} = 0$.

Cauchy Euler

$$x^2y'' + 7xy' + 25y = 0$$

$$m(m-1) + 7m + 25 = 0 \quad (4)$$

$$m^2 + 6m + 25 = 0$$

$$m = \frac{-6 \pm \sqrt{36-100}}{2}$$

$$= -3 \pm 4i$$

(2)

so $y = C_1 x^{-3} \cos 4\ln x$ (2)

$+ C_2 x^{-3} \sin 4\ln x$ (2)

10 Points

5. Find the inverse Laplace transform of $\frac{4s^2 - 22s}{(s^2+16)(s^2-14s+53)}$ by writing your answer in the form of a convolution (an unevaluated integral).

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+16} \cdot \frac{4s-22}{(s-7)^2+4}\right]$$

$\downarrow$

$\cos 4t$ (2)

$$\mathcal{L}^{-1}\left[\frac{4(s-7) + 28 - 22}{(s-7)^2+4}\right] \quad (3)$$

$$= e^{7t}\mathcal{L}^{-1}\left[\frac{4s}{s^2+4}\right] + e^{7t}\mathcal{L}^{-1}\left[\frac{2\cdot 3}{s^2+4}\right]$$

(2) (2)

$$\int_0^t \left[4e^{7\tau}\cos 2\tau + 3e^{7\tau}\sin 2\tau\right]\cos 4(t-\tau)\,d\tau$$

(4)

13 Points

35 Points

6. Find the general solution to $y'' + 2y' + y = \dfrac{e^{-x}}{1+x^2}$.

Var Param

$y'' + 2y' + y = 0$

$m^2 + 2m + 1$

$(m+1)^2 = 0 \quad m = \cancel{1}, -1$ (3)

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$ (2)

15 Points

$y = C_1 e^{-x} + C_2 x e^{-x}$

$y_p = V_1 e^{-x} + V_2 x e^{-x}$ (2)

$$V_1' = \frac{-xe^{-x} \cdot \frac{e^{-x}}{1+x^2}}{e^{-2x}} = \frac{-x}{1+x^2} \Rightarrow V_1 = \int -\frac{x}{1+x^2}\,dx = -\frac{1}{2}\ln|1+x^2|$$ (3)

$$V_2' = \frac{e^{-x}\,\frac{e^{-x}}{1+x^2}}{e^{-2x}} = \frac{1}{1+x^2} \Rightarrow V_2 = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x$$ (3)

$$y = C_1 e^{-x} + C_2 x e^{-x} + -\frac{1}{2}e^{-x}\ln|1+x^2| + x(\tan^{-1}x)e^{-x}$$

(2)

(6½)

7. Use the Laplace transform technique to solve $y'' + 12y' + 36y = f(t)$ given $y(0) = 1$, $y'(0) = -12$ and $f(t) = \begin{cases} 0, & 0 \le t < 4 \\ 6e^{-6t}, & t \ge 4 \end{cases}$. $= 0 + U(t-4)\,6e^{-6t}$ (2)

$$s^2 Y - s\overset{1}{y(0)} - \overset{-12}{y'(0)} + 12[sY - \overset{1}{y(0)}] + 36Y = 6e^{-4s}\mathcal{L}[e^{-6(t+4)}]$$

20 Points

$$Y\underbrace{[s^2+12s+36]}_{(s+6)^2} = s + \frac{6e^{-4s}e^{-24}}{s+6}$$ (2) (2) (4)

$$Y = \frac{s+6-6}{(s+6)^2} + \frac{6e^{-24}e^{-4s}}{(s+6)^3}$$

$$= \frac{1}{s+6} - \frac{6}{(s+6)^2} + U(t-4)\,6e^{-24}\mathcal{L}^{-1}\left[\frac{1}{(s+6)^3}\right]_{t \to t-4}$$

$$y(t) = e^{-6t} - 6e^{-6t}\cdot t + U(t-4)\,6e^{-24}e^{-6(t-4)}\frac{1}{2}\mathcal{L}^{-1}\left[\frac{t^2}{s^3}\right]$$

$$= e^{-6t} - 6te^{-6t} + 3U(t-4)\,e^{-24}e^{-6(t-4)}(t-4)^2$$

(3) (3) (4)

35 Points

8. Consider the system of differential equations $\frac{dx}{dt} = -2x - y$

$$\frac{dy}{dt} = x .$$

Write this system in matrix form and then use the matrix methods to find the general solution to this system.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} -2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 2\lambda + \lambda^2 + 1 = (\lambda+1)^2 \Rightarrow \lambda = -1, -1 \quad (4)$$

15 Points

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad k_1 = -k_2 \quad \text{evector } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3)$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad k_1 = -k_2 - 1 \quad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -k_2 - 1 \\ k_2 \end{pmatrix} = k_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (4)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t}\right]$$

(1) (2) (1)

$5\frac{1}{2}$

9. Find the general solution to $\frac{d^3y}{dx^3} + 9\frac{dy}{dx} = 36x + 24\sin x$.

$y''' + 9y' = 0$

$m^3 + 9m = 0$ $\qquad m's = 0, 0, \pm i$

$m = 0 \pm 3i$

$y_c = C_1 + C_2 \cos 3x + C_3 \sin 3x \quad (3)$

15 Points

$y = y_c + Ax + Bx^2 + C\cos x + D\sin x \quad (8)$

$y_p' = A + 2Bx - C\sin x + D\cos x$

$y_p'' = 2B - C\cos x - D\sin x$

$y_p''' = C\sin x - D\cos x$

$C\sin x - D\cos x + 9\left[A + 2Bx - C\sin x + D\cos x\right] = 36x + 24\sin x$

$9A = 0 \quad A = 0$

$18B = 36 \quad B = 2$ $\qquad (4)$

$C - 9C = -8C = 24 \quad C = -3$

$-D + 9D = 0 \Rightarrow D = 0$

$Y = C_1 + C_2\cos 3x + C_3\sin 3x + 2x^2 - 3\cos x$

30 Points

10. Find a series solution about x = 0 for the differential equation $(x^2+1)y'' - 6y = 0$.

$y = \sum_{n=0}^{\infty} a_n x^n$

(5)

$\sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=4}^{\infty} a_{n-2}(n-2)(n-3) x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 6\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$ (4)

1

$n=2 \quad a_2 \cdot 2 \cdot 1 \cdot x^0 - 6a_0 x^0 \Rightarrow$ so $a_2 = 3a_0$ (1)

$n=3 \quad a_3 \cdot 3 \cdot 2\, x^1 - 6a_1 x^1 \Rightarrow$ so $a_3 = a_1$ (1)

$n \geq 4 \quad a_{n-2}(n^2-5n+6) + a_n n(n-1) - 6a_{n-2} = 0$

so $a_n = \frac{5n-n^2}{n(n-1)} a_{n-2} = \frac{5-n}{n-1} a_{n-2} \quad n \geq 4$ (4)

$a_4 = \frac{+1}{3} a_2 \qquad a_5 = 0$

$a_6 = \frac{-1}{5} a_4 \qquad a_7 = \frac{2}{6} a_5 = 0$

$\vdots$

$a_8 = \frac{-3}{7} a_6 \qquad 0$

$a_{10} = \frac{-5}{9} a_8$ (1)

$\vdots$

$a_{2k} = \frac{5-2k}{2k-1} a_{2k-2}$

$\cancel{a_4 \cdot a_6 \cdot a_8 \cdots a_{2k-2}}\, a_{2k} = \frac{(1)(-1)(-3)\cancel{(-5)} \cdots (-1)(2k-5)\, a_2 \cancel{\cdot a_4 \cdot a_6 \cdots a_{2k-2}}}{\cancel{3 \cdot 5 \cdot 7 \cdot 9} \cdots (2k-3)(2k-1)}$

$a_{2k} = \frac{(-1)^{k-2}}{(2k-3)(2k-1)} a_2$ (4)

so $y = a_1[x + x^3]$ (2)

$+ a_0\left[1 + 3x^2 + \sum_{k=2}^{\infty} \frac{(-1)^{k-2} \cdot 3}{(2k-3)(2k-1)} x^{2k}\right]$

(3)

20 Points