Diff Eq - 3450:235
EXAM # 2 Spr 98

NAME Key
ROW

100 Points

Show **ALL** your work.

1. Find the general solution to $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 4x + e^x$.

15 Points

$y = y_c + y_p$ Und Coef.

$y = e^{mx} \Rightarrow m^4 - m^2 = 0 \Rightarrow m^2(m^2-1) = 0$ $m = 0, 0, 1, -1$

So $y_c = C_1 + C_2 x + C_3 e^x + C_4 e^{-x}$ (4)

m for $y_p \Rightarrow 0, 0, 1 \Rightarrow 0, 0, 1, -1, 0, 0, 1$

Guess $y_p = Ax^2 + Bxe^x + Cx^3$ (3)

$y_p' = 2Ax + B[xe^x + e^x] + 3Cx^2$

$y_p'' = 2A + B[xe^x + 2e^x] + 6Cx$

$y_p''' = B[xe^x + 3e^x] + 6C$

$y_p^{IV} = B[xe^x + 4e^x]$

Substitute $\Rightarrow B[\cancel{xe^x} + 4e^x] - 2A - B[\cancel{xe^x} + 2e^x] - 6Cx = 4x + e^x$

$\Rightarrow A = 0$, $-6C = 4 \Rightarrow C = -2/3$, $4B - 2B = 1 \Rightarrow B = 1/2$
(2) (2) (2)

So $y = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} + \frac{1}{2} x e^x - \frac{2}{3} x^3$
(2)

2. A 96 lb mass is attached to a spring hanging from the ceiling. This causes the spring to stretch 8 ft on coming to rest at equilibrium. The damping constant for the system is 18 $\frac{\text{lb} - \text{sec}}{\text{ft}}$.

At time t = 0 sec the mass is pulled down 6 inches below the equilibrium point and given an upward velocity of 4 ft/sec. The motion of the mass is further driven by an external force of 12 cos(2t) + 3 sin(2t) lbs. Write down the governing differential equation and initial conditions for the motion of the mass. DO NOT SOLVE THE EQUATION.

$F = kx \Rightarrow 96 = k \cdot 8 \Rightarrow k = 12$ lb/ft

$C = 18$

10 Points

$x(0) = 6/12$ ft (3)

$x'(0) = -4$ ft/sec

$\frac{96}{32} x'' + 18x' + 12x = [12\cos 2t + 3\sin 2t]$

(1) (1) (3) (2)

25 Points

3. Find the general solution to $3y'' + 27y = 18\csc(\overset{3}{\cancel{9}}x)$.

15 Points

Variation of parameters

$y = y_c + y_p$ $\quad y'' + 9y = 6\csc(3x)$

Get $y_c \Rightarrow 3m^2 + 27 = 0 \quad m^2 + 9 = 0 \Rightarrow m = \pm 3i$

So $y_c = C_1 \cos 3x + C_2 \sin 3x$ (4)

$y_p = V_1(x)\cos 3x + V_2(x)\sin 3x$ (2) $\quad W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$ (2)

$V_1' = \dfrac{-6\csc 3x \cdot \sin 3x}{3} = -2 \Rightarrow V_1 = -2x$ (3)

$V_2' = \dfrac{6\csc 3x \cdot \cos 3x}{3} = 2\dfrac{\cos 3x}{\sin 3x} \Rightarrow V_2 = \frac{2}{3}\ln|\sin 3x|$ (3)

So $y = C_1 \cos 3x + C_2 \sin 3x$

$\quad - 2x\cos 3x + \frac{2}{3}\ln|\sin 3x| \cdot \sin 3x$

(1)

4. One solution of the equation $4(x-3)^2y'' + 8(x-3)y' + y = 0$ is $y = (x-3)^{-1/2}$. Find the general solution to this same equation.

15 Points

$y'' + \dfrac{8}{4(x-3)}y' + \dfrac{y}{4(x-3)^2} = 0$ (3)

(4) $y_2 = y_1 \displaystyle\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx = (x-3)^{-1/2} \cdot \int \frac{e^{-\int \frac{2}{x-3}dx}}{(x-3)^{-1}}\,dx$ (1)

$= (x-3)^{-1/2} \displaystyle\int \frac{e^{-2\ln|x-3|}}{(x-3)^{-1}}\,dx = (x-3)^{-1/2}\int \frac{(x-3)^{-2}}{(x-3)^{-1}}\,dx$

$= (x-3)^{-1/2} \displaystyle\int \frac{1}{(x-3)}\,dx = (x-3)^{-1/2} \cdot \ln|x-3|$ (5)

$\therefore y = C_1(x-3)^{-1/2} + C_2(x-3)^{-1/2} \cdot \ln|x-3|$

(2)

30 Points

5. A solution to $x^2y'' - 3xy' + 3y = 2x^4e^x$ is $y = 2x^2e^x - 2xe^x$. Solve the following problem:
$x^2y'' - 3xy' + 3y = -6x^4e^x$, $y(1) = 0$, $y'(1) = 8e$.

$y = y_c + y_p \qquad y_p = -3\{2x^2e^x - 2xe^x\}$

15 Points

$x^2y'' - 3xy' + 3y = 0 \quad$ try x^m (2)

$\Rightarrow m(m-1) - 3m + 3 = m^2 - 4m + 3 = (m-3)(m-1) = 0 \Rightarrow$

$m = 1, 3$ (3)

(1) (5)

$So\ y = C_1 x + C_2 x^3 + -3[2x^2e^x - 2xe^x]$

$y(1) = C_1 + C_2 + -3[0] = 0 \Rightarrow C_1 = -C_2$

$y'(1) = 8e = C_1 + 3C_2 + -3[4xe^x + 2x^2e^x - 2e^x - 2xe^x]_{x=1}$

$8e = C_1 + 3C_2 - 3[2e^x] = C_1 + 3C_2 - 6e$

$so\ 14e = 2C_2 \Rightarrow C_2 = 7e, \quad C_1 = -7e$ (4)

6. Solve the system of equations: $\dfrac{dx}{dt} = \dfrac{1}{2}x + 9y$

$\dfrac{dy}{dt} = \dfrac{1}{2}x + 2y$

$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \frac{1}{2} & 9 \\ \frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

15 Points

Find e. values (9)

$\begin{vmatrix} \frac{1}{2}-\lambda & 9 \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = (\frac{1}{2}-\lambda)(2-\lambda) - \frac{9}{2} = 1 - \frac{5}{2}\lambda + \lambda^2 - \frac{9}{2} = 0$

$\lambda^2 - \frac{5}{2}\lambda - \frac{7}{2} = 0$

$2\lambda^2 - 5\lambda - 7 = (2\lambda - 7)(\lambda + 1) = 0$

$\lambda = \frac{7}{2}, -1$ (4)

$\frac{7}{2}$

$\begin{pmatrix} \cdot & 9 \\ \cdot & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3k_1 + 9k_2 = 0 \Rightarrow k_1 = 3k_2 \Rightarrow k_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (3)

-1

$\begin{pmatrix} \cdot & 9 \\ \cdot & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \frac{3}{2}k_1 + 9k_2 = 0 \Rightarrow k_1 = -6k_2 \Rightarrow k_2\begin{pmatrix} -6 \\ 1 \end{pmatrix}$ (3)

$\cdot) = C_1\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{\frac{7}{2}t} + C_2\begin{pmatrix} -6 \\ 1 \end{pmatrix}e^{-t}$ (4)

30 Points

7. A mass of M slugs $\left(\frac{\text{lb}-\text{sec}^2}{\text{ft}}\right)$ is attached to a spring with a spring constant of $6\ \frac{\text{lb}}{\text{ft}}$ and a damping constant of $\frac{1}{2}\ \frac{\text{lb}-\text{sec}}{\text{ft}}$. An external force of 10 cos(3t) lbs is applied to this system. The initial conditions for the motion x(t) of the mass are $x(0) = 2$, and $x'(0) = 0$. The equation of motion for the mass is found to be

$$x(t) = e^{-t/2}\left[\frac{-4}{3}\cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{64}{3\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right)\right] + \frac{10}{3}[\cos(3t) + \sin(3t)].$$

a) Find the numerical value of the mass M.

M = mass, $k = 6$, $c = 1/2$

$Mx'' + \frac{1}{2}x' + 6x = 10\cos 3t$

so $m = \dfrac{\frac{-\frac{1}{2}}{M} \pm \sqrt{\left(\frac{-1}{2m}\right)^2 - 4 \cdot \frac{6}{m}}}{2}$ 3

need $\frac{-1}{4M} = -\frac{1}{2} \Rightarrow M = \frac{1}{2}$ slug

2

5 Points

b) What value of the damping constant leads to critical damping?

Need $\left(\frac{c}{M}\right)^2 - \frac{24}{M} = 0$ 1

$\Rightarrow c^2 = 24M$

$c = \sqrt{24M} = \sqrt{12} = 3.464$

1

2 Points

c) Find the amplitude and phase shift of the steady-state motion of the mass.

steady state $= \frac{10}{3}[\cos 3t + \sin 3t]$ 2

Amplitude $= \sqrt{(10/3)^2 + (10/3)^2} = \frac{10}{3}\sqrt{2}$ 3 $= 4.714$

phase shift $\phi = \tan^{-1}\frac{\frac{10}{3}}{10/3} = \tan^{-1} 1 = \pi/4$ 3 $= 0.785$

8 Points

15 Points