100 Points

Diff Eq - 3450:335
EXAM #2 Fall00

NAME Key
ROW

Show **ALL** your work.

1. Find the general solution to $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 4\cos x + 12x$.

$y = y_c + y_p$ $\qquad$ m for $y_p = \pm i, 0, 0$

Get $y_c \Rightarrow y = e^{mx} \Rightarrow m^4 + m^2 = 0$

$m^2(m^2+1) = 0$

$m = 0, 0, \pm i$

$y_c = C_1 + C_2 x + C_3 \sin x + C_4 \cos x$ (4)

all m's $0, 0, \pm i, \pm i, 0, 0$

(8)

$y = C_1 + C_2 x + C_3 \sin x + C_4 \cos x + \underbrace{A x \sin x + Bx\cos x + Cx^2 + Dx^3}_{y_p}$

$y_p' = A[x\cos x + \sin x] + B[-x\sin x + \cos x] + 2Cx + 3Dx^2$

$y_p'' = A[-x\sin x + 2\cos x] + B[-x\cos x - 2\sin x] + 2C + 6Dx$

$y_p''' = A[-x\cos x - 3\sin x] + B[x\sin x - 3\cos x] + \quad 6D$

$y_p^{iv} = [A[x\sin x - 4\cos x] + B[x\cos x + 4\sin x]$

$y_p^{iv} + y_p'' = -2A\cos x + 2B\sin x + 2C + 6Dx = 4\cos x + 12x$

$-2A = 4$ so $A = -2$

$B = 0$ $\qquad$ (4)

$C = 0$

$6D = 12 \Rightarrow D = 2$

$y = C_1 + C_2 x + C_3 \sin x + C_4 \cos x - 2x\sin x + 2x^3$

16 Points

2. A solution to $4xy'' + 2y' + y = -3\sin\sqrt{x}$ is $y = \frac{3}{2}\sqrt{x}\cos\sqrt{x}$. A solution to $4xy'' + 2y' + y = 0$ is $y = \cos\sqrt{x}$. Find the general solution to $4xy'' + 2y' + y = \sin\sqrt{x}$.

$y = y_c + y_p \qquad y_c = C_1 \cos\sqrt{x} + C_2 y_2$

16 Points

$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$

$y_p = -\frac{1}{3}\left(\frac{3}{2}\sqrt{x}\cos\sqrt{x}\right)$

$= -\frac{1}{2}\sqrt{x}\cos\sqrt{x}$ (5)

$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx = \cos\sqrt{x} \int \frac{e^{-\int \frac{1}{2x} dx}}{\cos^2\sqrt{x}} = \cos\sqrt{x} \int \frac{e^{-\frac{1}{2}\ln x}}{\cos^2\sqrt{x}} dx$ (4)

$= \cos\sqrt{x} \int \frac{1/\sqrt{x}}{\cos^2\sqrt{x}} dx$ (4)

$= \cos\sqrt{x} \int \sec^2\sqrt{x} \frac{1}{\sqrt{x}} dx = \cos\sqrt{x} \cdot \tan\sqrt{x} \cdot 2 = 2\sin\sqrt{x}$

So $y = C_1 \cos\sqrt{x} + C_2 \sin\sqrt{x} - \frac{1}{2}\sqrt{x}\cos\sqrt{x}$ (3)

3. Solve the system of equations: $\frac{dx}{dt} = 2x - 4y$
$\frac{dy}{dt} = -x - y.$

$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

16 Points

Get e.values $\begin{vmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4 = -2 - 2\lambda + \lambda + \lambda^2 - 4$

$= \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$

$\lambda = 3, -2$ (4)

$\lambda = 3 \Rightarrow \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -k_1 - 4k_2 = 0$ or $k_1 = -4k_2 \Rightarrow \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ (3)

$\lambda = -2 \Rightarrow \begin{pmatrix} 4 & -4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -k_1 + k_2 = 0$ or $k_1 = k_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (3)

$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

(3) (3)

32 Points

4. Find the general solution to $y'' + 3y' + 2y = \frac{1}{1+e^x}$.

16 Points

$y = y_c + y_p \qquad e^{mx} \Rightarrow m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0 \quad m = -1, -2$

So $y_c = C_1 e^{-x} + C_2 e^{-2x}$ (4)

Guess $y_p = V_1 e^{-x} + V_2 e^{-2x}$ (4)

$w = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$ (2) $\quad V_1' = \frac{-\frac{1}{1+e^x} e^{-2x}}{-e^{-3x}} = \frac{e^x}{1+e^x}$

so $V_1 = \int \frac{e^x}{1+e^x} dx = \ln|1+e^x|$ (3)

$V_2' = \frac{\frac{1}{1+e^x} \cdot e^{-x}}{-e^{-3x}} = -\frac{e^{2x}}{1+e^x} = -e^x + \frac{e^x}{1+e^x}$ so $V_2 = \int \left(-e^x + \frac{e^x}{1+e^x}\right) dx$

$= -e^x + \ln|1+e^x|$ (3)

Thus

$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-x}\ln|1+e^x| + \underbrace{\cancel{-e^{-2x} e^{x}}}_{-e^{-x}} + e^{-2x}\ln|1+e^x|$

absorb into $C_1 e^{-x}$

5. Find the solution to $x^2y'' - 3xy' + 4y = 0$, subject to $y(1) = 2$ and $y'(1) = 3$.

16 Points

Cauchy Euler

$\Rightarrow y = x^m \Rightarrow m(m-1) + -3m + 4 = m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$y = C_1 x^2 + C_2 x^2 \ln x$

(6) (6)

$y(1) = 2 = C_1$

$y'(1) = C_1 \cdot 2 + \cancel{C_2 \cdot 2 \cdot \ln 1} + C_2 \cdot 1 = 3$

$= 4 + C_2 = 3 \Rightarrow C_2 = -1$

So $y = 2x^2 - x^2 \ln x$

(2) (2)

32 Points

6. A 64 lb weight is attached to a spring hanging from the ceiling. This causes the spring to stretch 0.519 ft on coming to rest at equilibrium. There is damping numerically equal to 1/5 the instantaneous velocity in this system.

Initially the weight is released 3 ft above the equilibrium position with an upward velocity of 40.64 ft/sec.

6a. Write down the governing differential equation and initial conditions for the motion of the weight. DO NOT SOLVE THE EQUATION.

$F = kx \Rightarrow 64 = k \cdot 0.519 \Rightarrow k = 123.31 \frac{lb}{ft}$

6 Points

$x(0) = -3$ (1) $\quad x'(0) = -40.64$ (1)

$2x'' + \frac{1}{5}x' + 123.31x = 0$

(1) (1) (2)

6b. The answer to the above differential equation is $x(t) = e^{-t/20}\left[-3\cos(7.85t) - 3\sqrt{3}\sin(7.85t)\right]$. Find all the times when the mass passes through the equilibrium position and indicate which times correspond to upward motion and which to downward motion.

$A = \sqrt{3^2 + 3^2 \cdot 3} = \sqrt{9+27} = \sqrt{36} = 6$

$\phi = \tan^{-1}\frac{3}{3\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \pi/6 + \pi = \frac{7\pi}{6}$ (2)

need third quadrant

6 Points

$\Rightarrow x = 6e^{-t/20}\left[\sin\left(7.85t + 7\pi/6\right)\right]$

$(=0 \Rightarrow 7.85t + 7\pi/6 = n\pi \Rightarrow t = \frac{n\pi - 7\pi/6}{7.85}$ (2) $\quad n = 2, 3, \ldots$

$n =$ even $\Rightarrow$ down (1) $\quad n =$ odd $\Rightarrow$ up (1)

6c. Find the time T, to within 1 second, such that $|x(t)| < 0.1$ ft for all $t > T$.

Need $\left|6e^{-t/20}\sin\left(7.85t + \frac{7\pi}{6}\right)\right| \leq 6e^{-t/20} < 0.1$ (2) (4)

8 Points

$\Rightarrow e^{-t/20} < \frac{0.1}{6}$

$-t/20 < \ln\frac{0.1}{6}$

or $t > -20\ln\frac{0.1}{6}$

$t > 81.9$ sec (2)

20 Points