Please leave this in the blue box!

100 Points

Diff Eq - 3450:235
EXAM #1 Spring 99

NAME______________________
ROW______________________

Show **ALL** your work.

1. Solve $x\frac{dy}{dx} + 4y = \frac{1}{x^4(x-1)}$. Linear

15 Points

$$\frac{dy}{dx} + \frac{4}{x}y = \frac{1}{x^5(x-1)} \quad 3$$

$$IF = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = x^4 \quad 4$$

$$\text{so } x^4\frac{dy}{dx} + 4x^3y = \frac{1}{x(x-1)}$$

$$\frac{d}{dx}\left(x^4 y\right) = \frac{-1}{x} + \frac{1}{x-1} \quad 2$$

$$\text{Thus } x^4 y = -\ln|x| + \ln|x-1| + c$$

4 2

or $y = \frac{1}{x^4}\left(-\ln|x| + \ln|x-1|\right) + \frac{c}{x^4}$

2a. A tank holds 400 gallons of water. Initially the tank is filled with 300 gallons of water with 50 kg of chlorine in solution. Water containing 2 kg of chlorine per gallon is entering the tank at the rate of 3 gal/min. The well-stirred mixture is flowing out of the tank at the rate of 1 gal/min. If C(t) denotes the amount of chlorine in kg in the tank at any time t in minutes, find a differential equation and initial condition which governs the amount of chlorine in the tank at any time t. DO NOT SOLVE THE EQUATION.

10 Points

3

$$\frac{dc}{dt} = 2\,\frac{kg}{gal}\cdot 3\,\frac{gal}{min} - \frac{c}{300+(3-1)t}\cdot 1\,\frac{gal}{min}$$

5

$$c(0) = 50$$

2

2b. Suppose that you know the solution C(t) to the above differential equation. How would you use this solution to find the amount of chlorine in the tank when the tank begins to overflow?

5 Points
30 Points

Overflow $\Rightarrow$ $400 = 300 + 2t$

$100 = 2t \Rightarrow t = 50$ min

4

Find $c(50)$

1

3. Solve $ydx - (x + \sqrt{xy})dy = 0$ subject to $y(0) = 1$.

15 Points

(4) Let $v = \frac{x}{y} \Rightarrow x = vy \quad dx = v\,dy + y\,dv$

$y(v\,dy + y\,dv) - (vy + \sqrt{vy^2})dy = 0$ (4) Homogeneous

$\cancel{yv\,dy} + y^2 dv - \cancel{vy\,dy} - y\sqrt{v}\,dy = 0$

$y\,dv = \sqrt{v}\,dy$

$\frac{dv}{\sqrt{v}} = \frac{dy}{y}$ (3)

$2\sqrt{v} = \ln|y| + c$

$2\sqrt{\frac{x}{y}} = \ln|y| + c$ (2)

so $y = Ce^{2\sqrt{\frac{x}{y}}}$

$y(0) = 1 = C$

Thus $y = e^{2\sqrt{\frac{x}{y}}}$

or $2\sqrt{\frac{x}{y}} = \ln|y|$ (2)

4. Solve $x\frac{dy}{dx} = 6x^2\sqrt{y} - 2y$.

15 Points

$x\frac{dy}{dx} + 2y = 6x^2 y^{1/2}$ Bernoulli

$y^{-1/2}\frac{dy}{dx} + \frac{2}{x}y^{1/2} = 6x$ (3)

let $z = y^{1/2}$ (3)

$dz/dx = \frac{1}{2}y^{-1/2}\,dy/dx$

So $2\frac{dz}{dx} + \frac{2}{x}z = 6x$

$\frac{dz}{dx} + \frac{1}{x}z = 3x$ (2)

$IF = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$ (2)

Thus

$x\frac{dz}{dx} + z = 3x^2$

$\frac{d}{dx}(xz) = 3x^2$

$xz = x^3 + C$ (4)

$z = x^2 + Cx^{-1}$

So $y^{1/2} = x^2 + Cx^{-1}$

(1)

30 Points

5. Use the substitution $y(x) = x + \frac{1}{v(x)}$ to solve the equation $\frac{dy}{dx} = 2x^2 + \frac{y}{x} - 2y^2$.

This substitution should result in a simpler equation for $v(x)$. Now solve for $v(x)$.

$$\frac{dy}{dx} = 1 + (-1)\,v^{-2}(x)\,\frac{dv}{dx} \qquad 3$$

$$\text{So} \quad 1 - v^{-2}\frac{dv}{dx} = 2x^2 + \frac{x + \frac{1}{v}}{x} - 2\left(x + \frac{1}{v}\right)^2$$

$$1 - v^{-2}\frac{dv}{dx} = 2x^2 + 1 + \frac{1}{xv} - 2\left(x^2 + \frac{2x}{v} + \frac{1}{v^2}\right)$$

Multiply by v^2

$$\cancel{v^2} - \frac{dv}{dx} = \cancel{2x^2v^2} + \cancel{v^2} + \frac{1}{x}v - \cancel{2x^2v^2} - 4xv - 2$$

$$\text{Thus} \quad \frac{dv}{dx} + \left(-4x + \frac{1}{x}\right)v = 2 \qquad \text{Linear} \qquad 4$$

$$IF = e^{\int\left(-4x + \frac{1}{x}\right)dx} = e^{-2x^2 + \ln|x|} = x e^{-2x^2} \qquad 3$$

$$\Rightarrow \quad x e^{-2x^2}\frac{dv}{dx} + \left(-4x^2 e^{-2x^2} + e^{-2x^2}\right)v = 2x e^{-2x^2}$$

$$\frac{d}{dx}\left(x e^{-2x^2} v\right) = 2x e^{-2x^2} \qquad 4$$

$$\Rightarrow \quad x e^{-2x^2} v = -\frac{1}{2} e^{-2x^2} + C$$

$$v = \frac{-1}{2x} + \frac{C}{x e^{-2x^2}} \qquad 1$$

$$\text{So} \quad y = x + \frac{1}{v} = x + \frac{1}{\frac{-1}{2x} + \frac{Ce^{2x^2}}{x}}$$

15 Points

6. A man has a fortune which he is spending at a rate that is proportional to the square of his present wealth. If he had \$1 million a year ago and has \$1/2 million today, how much will he be worth in 9 years?

15 Points

Let $A(t)$ = amount money (2)

(5) $\frac{dA}{dt} = kA^2$ $\quad A(0) = 1$, $A(1) = 1/2$ Find $A(10)$

$\frac{dA}{A^2} = k\,dt$

so $-\frac{1}{A} = kt + c$

or $A = \frac{-1}{kt+c}$ (3)

$A(0) = 1 = \frac{-1}{c} \Rightarrow c = -1$

$A(1) = \frac{1}{2} = \frac{-1}{k-1}$ (4)

so $k - 1 = -2 \Rightarrow k = -1$

thus

$A(t) = \frac{1}{t+1}$

$A(10) = \frac{1}{11}$ million dollars

(1)

7a. Suppose the temperature of a cup of coffee obeys Newton's Law of Cooling. The cup of coffee is initially at 95°C and cools to 80°C in 5 minutes while sitting in a room of temperature 21°C. If T(t) denotes the temperature of the coffee at any time t in minutes, find a differential equation and conditions which govern the temperature at any time t. DO NOT SOLVE THE DIFFERENTIAL EQUATION.

5 Points

$T(0) = 95$
$T(5) = 80$ (2)

$\frac{dT}{dt} = k(21 - T)$

(5)

7b. Suppose you know the solution $T(t) = 74\, e^{(1/5 \ln(59/74))t} + 21$ for the temperature of the coffee. When will the coffee be 50°C? What is the steady state temperature of the coffee?

5 Points

$50 = 74\, e^{\left(\frac{1}{5} \ln \frac{59}{74}\right)t} + 21$ (3)

$\Rightarrow \frac{29}{74} = e^{\left(1/5 \ln \frac{59}{74}\right)t}$ so $t = \frac{5 \ln \frac{29}{74}}{\ln 59/74} \sim 20.7$ min

Steady state $\Rightarrow T(t \to \infty) = 21$

(2)

25 Points