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| 100 Points |

Diff Eq - 3450:335
Exam #2 S06

NAME Key
ROW

Show **ALL** your work.

1. Find the general solution to $\frac{d^3y}{dx^3} - 9\frac{dy}{dx} = \underbrace{8x + 2e^{2x} + 15\sin x}_{m's\,=\,0,0,\,2,\,\pm i}$.

Get $y_c$ $\quad y''' - 9y' = 0$

$y = e^{mx} \Rightarrow m^3 - 9m = m(m^2-9) = 0 \Rightarrow m = 0, 3, -3$

so $y_c = C_1 + C_2 e^{3x} + C_3 e^{-3x}$ $\quad$ 3

List m's $\Rightarrow 0, 3, -3, 0, 0, 2, \pm i$ $\quad$ 2 $\quad$ 3

$$\text{so } y = \underbrace{C_1 + C_2 e^{3x} + C_3 e^{-3x}}_{y_c} + \underbrace{Ax + Bx^2 + Ce^{2x} + D\cos x + E\sin x}_{y_p}$$

1 $\quad$ 2

$y_p' = A + 2Bx + 2Ce^{2x} - D\sin x + E\cos x$

$y_p'' = 2B + 4Ce^{2x} - D\cos x - E\sin x$

$y_p''' = 8Ce^{2x} + D\sin x - E\cos x$

$$y''' - 9y' = 8Ce^{2x} + D\sin x - E\cos x$$
$$-9\left[A + 2Bx + 2Ce^{2x} - D\sin x + E\cos x\right] = 8x + 2e^{2x} + 15\sin x$$

$1: \quad -9A = 0 \Rightarrow A = 0$

$x: \quad -18B = 8 \Rightarrow B = -8/18 = -4/9$

$e^{2x}: \quad 8C - 18C = -10C = 2 \Rightarrow C = -1/5$ $\quad$ 5

$\sin x: \quad D + 9D = 10D = 15 \Rightarrow D = 15/10 = 3/2$

$\cos x: \quad -E - 9E = 0 \Rightarrow E = 0$

Final Ans

$$y = C_1 + C_2 e^{3x} + C_3 e^{-3x} - \frac{4}{9}x^2 - \frac{1}{5}e^{2x} + \frac{3}{2}\cos x$$

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| 16 Points |

2. A solution to $xy'' + 2(2x-1)y' + 4(x-1)y = 0$ is $y = e^{-2x}$. A solution to $xy'' + 2(2x-1)y' + 4(x-1)y = 4x^3 + 12x^4 + 4x^5$ is $y = x^4$. A solution to $xy'' + 2(2x-1)y' + 4(x-1)y = -2x + 4x^2 + 4x^3$ is $y = x^2$. Find the general solution to $xy'' + 2(2x-1)y' + 4(x-1)y = x - 2x^2 + 6x^4 + 2x^5$.

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$y = y_c + y_p$

Use red of order to get $y_2 = y_1 \int \frac{e^{-\int P\,dx}}{(y_1)^2}\,dx$

$$y_2 = e^{-2x} \int \frac{e^{-\int \frac{2(2x-1)}{x}dx}}{e^{-4x}}\,dx = e^{-2x}\int \frac{e^{-\int (4-\frac{2}{x})dx}}{e^{-4x}} = e^{-2x}\int \frac{e^{-4x}e^{2\ln x}}{e^{-4x}}\,dx$$

(1) (2) (1)

$$= e^{-2x}\int x^2\,dx = \frac{1}{3}x^3 e^{-2x}$$

(3)

so $y_c = C_1 e^{-2x} + C_2 x^3 e^{-2x}$

Use superposition to get $y_p$

$$x - 2x^2 + 6x^4 + 2x^5 = \frac{1}{2}(4x^3 + 12x^4 + 4x^5) + -\frac{1}{2}(-2x + 4x^2 + 4x^3)$$

so $y = C_1 e^{-2x} + C_2 x^3 e^{-2x} + \frac{1}{2}x^4 - \frac{1}{2}x^2$

(1) (2) (5) (3)

3. Find the general solution to $x^3y''' + 9x^2y'' + 65xy' = 0$.

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| 16 Points |

C-E so try $y = x^m$

$m(m-1)(m-2) + 9m(m-1) + 65m = 0$

$m[m^2 - 3m + 2 + 9m - 9 + 65] = 0$

$m[m^2 + 6m + 58] = 0$

$m = 0$, $m = \frac{-6 \pm \sqrt{36 - 232}}{2} = \frac{-6 \pm i\sqrt{196}}{2} = -3 \pm 7i$ (4)

(2)

so $y(x) = C_1 + C_2 x^{-3}\cos(7\ln x) + C_3 x^{-3}\sin(7\ln x)$

(2) (4) (4)

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| 32 Points |

4. Find the general solution to $x^2y'' + xy' + 4y = \sec(2\ln x)$. Note that using the substitution $u = \ln x$, one finds $\int \frac{\tan(2\ln x)}{x}dx = -\frac{1}{2}\ln|\cos(2\ln x)|$.

$y = y_c + y_p$

Solve $x^2y'' + xy' + 4y = 0$

$y = x^m \Rightarrow m(m-1) + m + 4 = 0$

$m^2 + 4 = 0$ so $m = \pm 2i$

So $y = C_1\cos(2\ln x) + C_2\sin(2\ln x)$ (4)

Get $y_p$ by var of parameters

$y_p = V_1\cos(2\ln x) + V_2\sin(2\ln x)$

$$W = \begin{vmatrix} \cos(2\ln x) & \sin(2\ln x) \\ -\frac{2}{x}\sin(2\ln x) & \frac{2}{x}\cos(2\ln x) \end{vmatrix} = \frac{2}{x}$$ (1)

$$V_1' = \frac{-\sin(2\ln x)\cdot\frac{\sec(2\ln x)}{x^2}}{2/x} = -\frac{1}{2}\frac{\tan(2\ln x)}{x}$$

so $V_1 = \frac{1}{4}\ln|\cos(2\ln x)|$ (3)

$$V_2' = \frac{\cos(2\ln x)\cdot\frac{\sec(2\ln x)}{x^2}}{\frac{2}{x}} = \frac{1}{2x}$$

so $V_2 = \frac{1}{2}\ln x$ (3)

Final Ans (2)

$$y = C_1\cos(2\ln x) + C_2\sin(2\ln x) + \frac{1}{4}\ln|\cos(2\ln x)|\cdot\cos(2\ln x) + \frac{1}{2}\ln x\sin(2\ln x)$$ (3)

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| 16 Points |

5. Solve the system of equations:

$$\frac{dx}{dt} = 12x - 17y$$
$$\frac{dy}{dt} = 4x - 4y.$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 12-\lambda & -17 \\ 4 & -4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Get e-values

$$\begin{vmatrix} 12-\lambda & -17 \\ 4 & -4-\lambda \end{vmatrix} = (12-\lambda)(-4-\lambda) + 68 = \lambda^2 - 8\lambda - 48 + 68 = 0$$

$\lambda^2 - 8\lambda + 20 = 0$ (3)

$$\lambda = \frac{8 \pm \sqrt{64-80}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

Get e-vector for $\lambda = 4+2i$:

$$\begin{pmatrix} 8-2i & -17 \\ 4 & -8-2i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (8-2i)k_1 - 17k_2 = 0$$

so $k_2 = \frac{8-2i}{17}k_1$

$$\begin{pmatrix} k_1 \\ \frac{8-2i}{17}k_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 17 \\ 8 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$ (5)

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1e^{4t}\left[\begin{pmatrix} 17 \\ 8 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right]$$ (4)

$$+ C_2e^{4t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 17 \\ 8 \end{pmatrix}\sin(2t)\right]$$ (4)

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| 16 Points |

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| 32 Points |

6. A 32 lb weight stretches a spring 32/5 ft. This weight is then lifted 5 ft above equilibrium and given a downward velocity of 29 ft/s. There is damping in this system that is equal to twice the instantaneous velocity.

6a. Write down the governing differential equation and initial conditions for the motion of the weight. DO NOT SOLVE THE EQUATION.

$32 = k \cdot 32/5 \Rightarrow k = 5 \text{ lb/ft}$

(1) $\frac{32}{32} x'' + 2x' + 5x = 0$ (1) (2)

$x(0) = -5$

$x'(0) = 29$ (2)

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| 6 Points |

6b. The answer to the above differential equation is $x(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. Find $c_1$ and $c_2$.

$x(0) = -5 = c_1$

$x' = c_1\left[-e^{-t}\cos 2t - e^{-t}\cdot 2\sin 2t\right] + c_2\left[-e^{-t}\sin 2t + e^{-t}2\cos 2t\right]$

$x'(0) = 29 = -5(-1) + c_2 \cdot 2 \Rightarrow 2c_2 = 24 \Rightarrow c_2 = 12$

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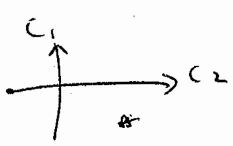
| 2 Points |

6c. Determine the first two times when the weight passes through the equilibrium position. For each time state whether the time corresponds to motion up or down.

$x(t) = e^{-t}\sqrt{(-5)^2 + 12^2}\,\sin(2t + \phi)$

$\phi = \tan^{-1}\frac{-5}{12} = -0.395$ (3)




$\sin(2t - 0.395) = \begin{matrix} 0 + 2n\pi \\ \pi + 2n\pi \end{matrix}$

$\Rightarrow t = \frac{0.395}{2} + n\pi \quad n = 0, 1, \ldots$ (2)

$\frac{\pi + 0.395}{2} + n\pi \quad n = 0, 1, \ldots$ (2)

$n = 0$

$\frac{0.395}{2} = 0.1975$ down (1)

$\frac{\pi + 0.395}{2} = 1.768$ up (1)

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| 9 Points |

6d. SET UP BUT DO NOT SOLVE the mathematical expression you would use to approximate the time T such that $|x(t)| < 0.01$ for all $t > T$.

$e^{-t}\sqrt{(-5)^2 + (12)^2} < 0.01$ (2) (1)

Solve for $t$

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| 3 Points |

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| 20 Points |