On Huppert’s Conjecture for Alternating Groups of Low Degrees

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Abstract. Bertram Huppert conjectured in the late 1990s that the nonabelian simple groups are determined up to an abelian direct factor by the set of their character degrees. Although the conjecture has been established for various simple groups of Lie type and simple sporadic groups, it is expected to be difficult for alternating groups. In [5], Huppert verified the conjecture for the simple alternating groups $A_n$ of degree up to 11. In this paper, we continue his work and verify the conjecture for the alternating groups of degrees 12 and 13.

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1 Introduction

It is known that, in general, the structure of a finite group is not determined entirely by the set of its irreducible character degrees. Among many examples, the non-isomorphic groups $D_8$ and $Q_8$ are perhaps the most well known and interesting as they have not only the same set of character degrees but also the same character table. However, it is believed that for some special families of groups, the set of character degrees determines them uniquely up to an abelian direct factor.

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Bertram Huppert conjectured in the late 1990s that the nonabelian simple groups are essentially determined by the set of their character degrees. In [4], he posed the following:

**Huppert’s Conjecture.** Let $G$ be a finite group and $H$ a finite nonabelian simple group such that the sets of character degrees of $G$ and $H$ are the same. Then $G \cong H \times A$, where $A$ is an abelian group.

Huppert [4] verified the conjecture on a case-by-case basis for many nonabelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type. There has been much recent work on the conjecture for the simple groups of Lie type, especially for those groups of rank one and two (see [8, 9, 10, 11]).

Included in a collection of preprints, Huppert [5] verified the conjecture for the alternating groups $A_n$ with $5 \leq n \leq 11$. In general, it is difficult to establish the conjecture for alternating groups. This is due to the arithmetical complexity of character degrees of $A_n$ and the fact that $A_n$ has many subgroups of small index. In this paper, we continue Huppert’s work and verify the conjecture for the alternating groups of degrees 12 and 13.

**Theorem 1.1.** Let $G$ be a finite group and $H$ be the alternating group of degree 12 or 13 such that the sets of character degrees of $G$ and $H$ are the same. Then $G \cong H \times A$, where $A$ is an abelian group.

Our proof follows the method of Huppert. In Step 1, we prove that the group $G$ is quasi-perfect, i.e., $G' = G''$. We then prove in Step 2 that if $G'/M$ is a chief factor of $G$, then $G'/M \cong H$. Next, we prove in Step 3 a technical result that every linear character of $M$ is stable under $G'$. Using this, it is then deduced in Step 4 that $G' \cong H$ and finally $G = G' \times C_G(G')$, which implies the conjecture since $C_G(G') \cong G/G'$ is abelian. We remark that Step 3 is the most difficult one and the proof of the other steps for the alternating groups of low degrees are fairly similar. Our proof of the stability of linear characters of $M$ under $G'$ indeed requires some new techniques involving the analysis of irreducible characters of various subnormal subgroups and subquotients inside the alternating groups.

**Notation.** Our notation and terminology are fairly standard (see, e.g. [2] and [6]). In particular, if $G$ is a finite group, then $\operatorname{Irr}(G)$ denotes the set of irreducible characters of $G$ and $\operatorname{cd}(G)$ denotes the set of degrees of characters in $\operatorname{Irr}(G)$. If $N \leq G$ and $\lambda \in \operatorname{Irr}(N)$, the induction of $\lambda$ from $N$ to $G$ is denoted by $\lambda^G$ and the set of irreducible constituents of $\lambda^G$ is denoted by $\operatorname{Irr}(G|\lambda)$. Furthermore, if $\chi \in \operatorname{Irr}(G)$, then $\chi_N$ is the restriction of $\chi$ to $N$. Finally, the least common multiple of two positive integers $a$ and $b$ is denoted by $\operatorname{lcm}(a, b)$.

2 Preliminaries

We present some known results from literature that will be needed in the proof of the main theorem.
Lemma 2.1. [6, Corollary 11.29] Let $N \trianglelefteq G$ and $\chi \in \text{Irr}(G)$. If $\psi$ is a constituent of $\chi_N$, then $\chi(1)\psi(1)$ divides $|G : N|$. 

Lemma 2.2. (Gallagher) Let $N \trianglelefteq G$ and $\chi \in \text{Irr}(G)$. If $\chi_N \in \text{Irr}(N)$, then $\chi \theta \in \text{Irr}(G)$ for every $\theta \in \text{Irr}(G/N)$. 

Lemma 2.3. [6, Lemma 12.3 and Theorem 12.4] Let $N \triangleleft G$ be maximal such that $G/N$ is solvable and nonabelian. Then one of the following holds:

(i) $G/N$ is a $p$-group for some prime $p$. If $\chi \in \text{Irr}(G)$ and $p \nmid \chi(1)$, then $\chi \tau \in \text{Irr}(G)$ for all $\tau \in \text{Irr}(G/N)$.

(ii) $G/N$ is a Frobenius group with an elementary abelian Frobenius kernel $F/N$. Thus, $[G : F] \in \text{cd}(G)$ and $|F : N| = p^a$, where $a$ is the smallest integer such that $|G : F|$ divides $p^a - 1$. For every $\psi \in \text{Irr}(F)$, either $[G : F]\psi(1) \in \text{cd}(G)$ or $|F : N|$ divides $\psi(1)^2$. If no proper multiple of $|G : F|$ is in $\text{cd}(G)$, then $\chi(1)$ divides $|G : F|$ for all $\chi \in \text{Irr}(G)$ such that $p \nmid \chi(1)$.

Lemma 2.4. In the context of (ii) in the above lemma, if $\chi \in \text{Irr}(G)$ so that $\text{lcm}(\chi(1), |G : F|)$ does not divide any character degree of $G$, then $p^a \nmid \chi(1)^2$.

Proof. Suppose that $\psi$ is a constituent of $\chi_F$. By Lemma 2.1, $\chi(1)/\psi(1)$ divides $|G : F|$. In other words, $\chi(1)$ divides $|G : F|\psi(1)$. Since $\text{lcm}(\chi(1), |G : F|)$ does not divide any character degree of $G$, we see that $|G : F|\psi(1)$ is not a degree of $G$. This forces $p^a \nmid \psi(1)^2$ by Lemma 2.3(ii). Since $\psi(1) | \chi(1)$, we have $p^a \nmid \chi(1)^2$. $\Box$

Lemma 2.5. [1, Lemma 5] Let $N = S \times \cdots \times S$, a direct product of $k$ copies of a nonabelian simple group $S$, be a minimal normal subgroup of $K$. If $\chi \in \text{Irr}(S)$ extends to $\text{Aut}(S)$, then $\chi(1)^k$ is a character degree of $K$.

Proof. Note that $N$ can be considered as a subgroup of $K/C_K(N)$ and $K/C_K(N)$ is embedded in $\text{Aut}(N) = \text{Aut}(S) \wr S_k$. Let $\lambda$ be an extension of $\chi$ to $\text{Aut}(S)$. Since $\lambda \times \cdots \times \lambda$ is invariant under $\text{Aut}(S) \wr S_k$, it is extendable to $\text{Aut}(S) \wr S_k$. Hence, the character $\chi \times \cdots \times \chi \in \text{Irr}(N)$ is extendable to $\text{Aut}(S) \wr S_k$. In particular, it can be extended to $K/C_K(N)$. The lemma follows. $\Box$

Lemma 2.6. [7, Theorem 2.8] Let $N$ be a normal subgroup of a group $G$ and let $\theta \in \text{Irr}(N)$ be $G$-invariant. If $\chi(1)/\theta(1)$ is a power of a fixed prime $p$ for every $\chi \in \text{Irr}(G/\theta)$, then $G/N$ is solvable.

Lemma 2.7. [4, Lemma 3] Let $M \trianglelefteq G$ and let $\theta \in \text{Irr}(M)$ be $G$-invariant. If $\phi \in \text{Irr}(G)$ lying above $\theta$, then $\phi = \theta_0\tau$, where $\theta_0$ is a character of an irreducible projective representative of $G$ of degree $\theta(1)$ and $\tau$ is a character of an irreducible projective representation of $G/M$.

Lemma 2.8. [4, Lemma 6] Suppose $M \trianglelefteq G' = G''$ and $\theta^g = \theta$ for all $g \in G'$ and $\theta \in \text{Irr}(M)$ such that $\theta(1) = 1$. Then $M' = [M, G']$ and $|M : M'|$ divides the order of the Schur multiplier of $G'/M$.

To end this section, we collect the information regarding the character degrees and maximal subgroups of the alternating groups $A_{12}$ and $A_{13}$ available in [2].
Table 1. Nontrivial character degrees of $A_{12}$

<table>
<thead>
<tr>
<th>Degree</th>
<th>(2^3\cdot3^3)</th>
<th>(2^3\cdot3^5)</th>
<th>(2^3\cdot3^7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>891</td>
<td>3\cdot11</td>
<td>2376</td>
</tr>
<tr>
<td>54</td>
<td>945</td>
<td>3\cdot5\cdot7</td>
<td>2673</td>
</tr>
<tr>
<td>55</td>
<td>1050</td>
<td>2\cdot3\cdot5\cdot7</td>
<td>2970</td>
</tr>
<tr>
<td>132</td>
<td>1155</td>
<td>3\cdot5\cdot7\cdot11</td>
<td>3080</td>
</tr>
<tr>
<td>154</td>
<td>1320</td>
<td>2^3\cdot3\cdot5\cdot11</td>
<td>3520</td>
</tr>
<tr>
<td>165</td>
<td>1408</td>
<td>2^7\cdot11</td>
<td>3564</td>
</tr>
<tr>
<td>275</td>
<td>1485</td>
<td>3^3\cdot5\cdot11</td>
<td>3696</td>
</tr>
<tr>
<td>297</td>
<td>1650</td>
<td>2\cdot3\cdot5^2\cdot11</td>
<td>3850</td>
</tr>
<tr>
<td>320</td>
<td>1728</td>
<td>2^6\cdot3^3</td>
<td>4158</td>
</tr>
<tr>
<td>330</td>
<td>1925</td>
<td>5^2\cdot7\cdot11</td>
<td>4455</td>
</tr>
<tr>
<td>462</td>
<td>2079</td>
<td>3^3\cdot7\cdot11</td>
<td>5632</td>
</tr>
<tr>
<td>616</td>
<td>2112</td>
<td>2^6\cdot3\cdot11</td>
<td>5775</td>
</tr>
</tbody>
</table>

Table 2. Maximal subgroups of $A_{12}$ and their indices

<table>
<thead>
<tr>
<th>Group</th>
<th>Indices</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11}$</td>
<td>12 = $2^2\cdot3$</td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>66 = $2\cdot3\cdot11$</td>
</tr>
<tr>
<td>$(A_6 \times 3)$ : 2</td>
<td>220 = $2^2\cdot5\cdot11$</td>
</tr>
<tr>
<td>$(A_6 \times A_6)$ : $2^2$</td>
<td>462 = $2\cdot3\cdot7\cdot11$</td>
</tr>
<tr>
<td>$(A_8 \times A_4)$ : 2</td>
<td>495 = $3^2\cdot5\cdot11$</td>
</tr>
<tr>
<td>$(A_7 \times A_5)$ : 2</td>
<td>792 = $2^3\cdot3^2\cdot11$</td>
</tr>
<tr>
<td>$M_{12}$</td>
<td>2520 = $2^3\cdot3^2\cdot5\cdot7$</td>
</tr>
<tr>
<td>$2^6 : 3^3$ : $S_4$</td>
<td>5775 = $3\cdot5^2\cdot7\cdot11$</td>
</tr>
<tr>
<td>$2^5 : S_6$</td>
<td>10395 = $3^3\cdot5\cdot7\cdot11$</td>
</tr>
<tr>
<td>$3^4 : 2^3 \cdot S_4$</td>
<td>15400 = $2^3\cdot5^2\cdot7\cdot11$</td>
</tr>
</tbody>
</table>

Table 3. Character degrees of $A_{13}$

<table>
<thead>
<tr>
<th>Degree</th>
<th>(2^3\cdot3^2\cdot11\cdot13)</th>
<th>(3^3\cdot7\cdot13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2574</td>
<td>7371</td>
</tr>
<tr>
<td>12</td>
<td>2860</td>
<td>7800</td>
</tr>
<tr>
<td>65</td>
<td>3003</td>
<td>8008</td>
</tr>
<tr>
<td>66</td>
<td>3432</td>
<td>8580</td>
</tr>
<tr>
<td>208</td>
<td>3575</td>
<td>9009</td>
</tr>
<tr>
<td>220</td>
<td>3640</td>
<td>9360</td>
</tr>
<tr>
<td>429</td>
<td>4004</td>
<td>10296</td>
</tr>
<tr>
<td>462</td>
<td>4212</td>
<td>11440</td>
</tr>
<tr>
<td>495</td>
<td>4290</td>
<td>11583</td>
</tr>
<tr>
<td>572</td>
<td>5005</td>
<td>12012</td>
</tr>
<tr>
<td>792</td>
<td>5148</td>
<td>12870</td>
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<tr>
<td>936</td>
<td>5720</td>
<td>15015</td>
</tr>
<tr>
<td>1287</td>
<td>6006</td>
<td>17160</td>
</tr>
<tr>
<td>1365</td>
<td>6435</td>
<td>20592</td>
</tr>
<tr>
<td>1430</td>
<td>6864</td>
<td>21450</td>
</tr>
</tbody>
</table>

3 Step 1: $G' = G''$

In this section, we will prove that $G' = G''$. Assume by contradiction that $G' \neq G''$. Let $N < G$ be maximal such that $G/N$ is solvable and nonabelian. By Lemma 2.3, we have two cases.
This case, and \( \text{lcm}(\text{by all these primes, Lemma 2.4 implies that } p \mid \theta(1) = p^b > 1. \) Inspecting the character degrees of \( A_{12} \), we see that it has only one degree which is a prime power and this degree is 11. Therefore, \( p = \theta(1) = 11 \). Take \( \chi \in \text{Irr}(G) \) with \( \chi(1) = 54 \). Then \( 11 \nmid \chi(1) \) and therefore Lemma 2.1 implies that \( \chi_N \in \text{Irr}(N) \). It follows by Lemma 2.2 that \( \chi \theta \notin \text{Irr}(G) \). However, \( \chi(1)\theta(1) = 54 \cdot 11 = 594 \) is not a character degree of \( A_{12} \), a contradiction.

Case 2. \( G/N \) is a Frobenius group with an elementary abelian Frobenius kernel \( F/N \). Thus, \( |G : F| \in \text{cd}(G) \) and \( |F : N| = p^a \), where \( a \) is the smallest integer such that \( |G : F| \) divides \( p^a - 1 \). For every \( \psi \in \text{Irr}(F) \), either \( |G : F|\psi(1) \in \text{cd}(G) \) or \( |F : N| \) divides \( \psi(1)^2 \).

If no proper multiple of \( |G : F| \) is in \( \text{cd}(G) \), then \( \chi(1) \) divides \( |G : F| \) for all \( \chi \in \text{Irr}(G) \) such that \( p \nmid \chi(1) \). It is routine to check that for any prime \( p \), we can find two different degrees \( d_1, d_2 \) of \( G \) which are coprime to \( p \) so that \( \text{lcm}(d_1, d_2) \) does not divide any degree of \( G \). This leads to a contradiction since \( d_1 \) and \( d_2 \) divide \( |G : F| \) and \( |G : F| \in \text{cd}(G) \).

Thus, we can assume that there is a proper multiple of \( |G : F| \) in \( \text{cd}(G) \). Recall that \( |G : F| \in \text{cd}(G) \). We consider two subcases depending on whether \( |G : F| \) is even or odd.

(a) \( 2 \) divides \( |G : F| \). Let \( \chi \) be the character of \( G \) of degree 1155 = 3 \cdot 5 \cdot 7 \cdot 11. Then \( \text{lcm}(|G : F|, \chi(1)) \) is divisible by 2, 3, 5, 7, 11. Since \( \text{cd}(G) \) has no degree divisible by all these primes, Lemma 2.4 implies that \( p^a \mid \chi(1)^2 \). Therefore, \( p = 3, 5, 7, \) or 11 and \( a \leq 2 \). Recalling from Lemma 2.3(ii) that \( |G : F| \) divides \( p^a - 1 \), we must have \( |G : F| \leq 11^2 - 1 \). This means that the only choice for \( |G : F| \) is 54. But even in this case, \( |G : F| \) does not divide \( p^a - 1 \).

(b) \( 2 \) does not divide \( |G : F| \). From the list of degrees of \( G \), we see that 11 divides \( |G : F| \). Let \( \chi \) be the character of \( G \) of degree 1050 = 2 \cdot 3 \cdot 5^2 \cdot 7. As in (a), \( p^a \mid \chi(1)^2 \). Hence, either \( p = 2, 3, 7 \) and \( a \leq 2 \) or \( p = 5 \) and \( a \leq 4 \). A routine check gives \( 11 \nmid (p^a - 1) \), which is a contradiction since 11 divides \( |G : F| \) and \( |G : F| \) divides \( p^a - 1 \).

The alternating group \( A_{13} \).

Case 1. \( G/N \) is a \( p \)-group for some prime \( p \). Since \( G/N \) is nonabelian, there is \( \theta \in \text{Irr}(G/N) \) such that \( \theta(1) = p^b > 1 \). On the other hand, \( A_{13} \) has no character of prime power degree by Table 3. Therefore, this case does not happen.

### Table 4. Maximal subgroups of \( A_{13} \) and their indices

<table>
<thead>
<tr>
<th>( A_{12} )</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{11} )</td>
<td>78 = 2 \cdot 3 \cdot 13</td>
</tr>
<tr>
<td>( (A_{10} \times 3) : 2 )</td>
<td>286 = 2 \cdot 11 \cdot 13</td>
</tr>
<tr>
<td>( (A_6 \times A_4) : 2 )</td>
<td>715 = 5 \cdot 11 \cdot 13</td>
</tr>
<tr>
<td>( (A_6 \times A_3) : 2 )</td>
<td>1287 = 3^2 \cdot 11 \cdot 13</td>
</tr>
<tr>
<td>( (A_5 \times A_6) : 2 )</td>
<td>1716 = 2^2 \cdot 3 \cdot 11 \cdot 13</td>
</tr>
<tr>
<td>( \text{PSL}_4(3) )</td>
<td>554400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11</td>
</tr>
<tr>
<td>13 : 6</td>
<td>39916800 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11</td>
</tr>
</tbody>
</table>
Case 2. $G/N$ is a Frobenius group with an elementary abelian Frobenius kernel $F/N$. As above, we consider the following three subcases.

(a) 2 divides $|G : F|$. Let $\chi$ be the character of $G$ of degree $1365 = 3 \cdot 5 \cdot 7 \cdot 13$. Then \( \text{gcd}((|G : F|, \chi(1))) \) is divisible by $2$, $3$, $5$, $7$, $13$. Since $\text{gcd}(G)$ has no degree divisible by all these primes, Lemma 2.4 implies that $p^n | \chi(1)^2$. Therefore, $p \leq 13$ and $a \leq 2$. Recalling from Lemma 2.3(ii) that $|G : F|$ divides $p^n - 1$, we must have $|G : F| \leq 13^2 - 1 = 168$. Table 3 then implies that $|G : F| = 12$ or $66$. The case $|G : F| = 66$ does not happen since $66 \nmid (p^n - 1)$ for any prime $p \leq 13$ and $a \leq 2$. So $|G : F| = 12$ and hence $p = 5, 7, \text{or } 13$.

If $p = 13$, then by taking $\psi(1) = 3^2 \cdot 5 \cdot 11$, we see that $\text{lcm}(\psi(1), |G : F|) = 2^2 \cdot 3^2 \cdot 5 \cdot 11$, which violates Lemma 2.4. If $p = 7$ or $5$, similar arguments lead to a contradiction if we take $\psi(1) = 5^2 \cdot 11 - 13$ or $3^4 \cdot 7 \cdot 13$, respectively.

(b) $5$ divides $|G : F|$. Let $\chi$ be a character of $G$ of degree $462 = 2 \cdot 3 \cdot 7 \cdot 11$. As in case (a), we obtain $p^n | \chi(1)^2$. It follows that $p \leq 11$ and $a \leq 2$ and therefore $|G : F| \leq p^n - 1 \leq 11^2 - 1 = 120$. This implies that $|G : F| = 65$, which is again a contradiction since $65 \nmid (p^n - 1)$ for any $p \leq 11$ and $a \leq 2$.

(c) Both $2$ and $5$ do not divide $|G : F|$. Examining the degrees of $G$, we see that $3 \cdot 13$ divides $|G : F|$. Let $\chi$ be the character of $G$ of degree $3640 = 2^3 \cdot 5 \cdot 7 \cdot 13$. Then $\text{lcm}((|G : F|, \chi(1)))$ is divisible by $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. Since $\text{gcd}(G)$ has no degree divisible by this number, Lemma 2.4 implies that $p^n | \chi(1)^2$. Therefore, $p^n \leq 13^2 = 169$. Using $|G : F| \leq p^n - 1$, we obtain $|G : F| \leq 168$. This is a contradiction since $G$ has no degree divisible by $3 \cdot 13$ and smaller than $169$.

4 Step 2: If $G' / M$ is a chief factor of $G'$, then $G' / M \cong H$.

Lemma 4.1. Let $K$ be a group. Suppose that $\text{cd}(K) \subseteq \text{cd}(A_{12})$ and $S \times \cdots \times S$, a direct product of $k$ copies of a nonabelian simple group $S$, is a minimal normal subgroup of $K$. Then $k = 1$ and $S \cong A_{12}$.

Proof. We will repeatedly use Lemma 2.5 in the proof. First, we assume that $S$ is a sporadic simple group or the Tits group. Inspecting [2], we can find many $\chi \in \text{Irr}(S)$ which are extendable to Aut$(S)$ but $\chi(1)^k \notin \text{cd}(A_{12})$ for any positive integer $k$. This violates Lemma 2.5.

Next, we consider the case where $S$ is a simple group of Lie type. Let $St$ denote the Steinberg character of $S$. It is well known (see [3] for instance) that $St$ is extendable to Aut$(S)$ and $St(1) = |S|_p$, the $p$-part of $|S|$. Therefore, $|St|^k \in \text{cd}(A_{12})$. This means $|St|^k = 11$, and hence $k = 1$ and $S = \text{PSL}_2(11)$. Taking $\chi \in \text{Irr}(\text{PSL}_2(11))$ of degree 10, we see that $\chi$ is extendable to Aut$(\text{PSL}_2(11))$ and therefore $10 \in \text{cd}(K)$, which leads to a contradiction.

Finally, assume that $S$ is the alternating group $A_n$ with $n \geq 5$. Note that $A_n$ has an irreducible character $\chi$ of degree $n - 1$ which corresponds to the partition $(2, 1^{n-2})$. Since this partition is not self-conjugate, $\chi_{A_n} \in \text{Irr}(A_n)$. Lemma 2.5 then implies that $(n - 1)^k \in \text{cd}(A_{12})$. If $k \geq 2$, then $(n - 1)^k = 2^6 \cdot 3^2$, which implies that $k = 2$ and $n = 25$. Since $A_{25}$ has the degree 11 of $25$ which extends to $S_{25}$ but $(11 \cdot 25)^2 \notin \text{cd}(A_{12})$, we get a contradiction.

Thus, $k = 1$ and hence $n - 1 \notin \text{cd}(A_{12})$. Noticing that every degree of $S$ divides a degree of $K$, we deduce that 13 does not divide any degree of $S$. The Ito-Michler
theorem then implies that 13 does not divide $|S| = |A_n|$. So $n \leq 12$. This and the fact $n - 1 \in \text{cd}(A_{12})$ imply that $n = 12$, as desired. □

**Lemma 4.2.** Let $K$ be a group and suppose that $\text{cd}(K) \subseteq \text{cd}(A_{13})$ and $S \times \cdots \times S$, a direct product of $k$ copies of a nonabelian simple group $S$, is a minimal normal subgroup of $K$. Then $k = 1$ and $S \cong A_{13}$.

**Proof.** The alternating groups, sporadic simple groups, and the Tits groups can be treated as in Lemma 4.1. So it remains to eliminate the case where $S$ is a simple group of Lie type. Let $\text{St}$ denote the Steinberg character of $S$. Then $\text{St}$ is extendable to $\text{Aut}(S)$ and $\text{St}(1) = |S|^p$, the $p$-part of $|S|$. Therefore, $|S|^k \in \text{cd}(A_{13})$, which leads to a contradiction since $\text{cd}(A_{13})$ does not have any prime power degree. □

As $G'$ is perfect by Step 1, we have $G'/M \cong S \times \cdots \times S$, a direct product of $k$ copies of a nonabelian simple group $S$, if $G'/M$ is a chief factor of $G$. Since $G'/M$ is a minimal normal subgroup of $G/M$ and $\text{cd}(G/M) \subseteq \text{cd}(G) = \text{cd}(H)$, Lemmas 4.1 and 4.2 then imply that $k = 1$ and $S \cong H$, as wanted.

**5 Step 3: Stability of the linear characters of $M$ under $G'$**

In this step, we prove that if $\theta \in \text{Irr}(M)$ and $\theta(1) = 1$, then $I := I_{G'}(\theta) = G'$. Assume by contradiction that $I < G'$, then $I \leq U < G'$ for some maximal subgroup $U$ of $G'$. Suppose

$$\theta^I = \sum_i \phi_i,$$

where $\phi_i \in \text{Irr}(I)$.

We then have $\phi^G_i \in \text{Irr}(G')$ and hence $\phi_i(1)|G': I| \in \text{cd}(G')$. It follows that

$$\phi_i(1)|G': U| \cdot |U : I| \text{ divides some degree of } G.$$

In particular, the index of $U/M$ in $G'/M$ divides some degree of $G$.

- **The alternating group $A_{12}$**. When $H = A_{12}$, we know that $G'/M \cong A_{12}$ by Step 2 and hence $U/M$ is a maximal subgroup of $A_{12}$. Inspecting Tables 1 and 2, we come up with the following cases.

  **Case 1.** $|G': U| = 12$ and $U/M \cong A_{11}$. Then $12|U : I|\phi_i(1)$ divides some degree of $G$. Since

$$|U : I| = \left| \frac{U}{M} : \frac{I}{M} \right|$$

is an index of a subgroup of $U/M \cong A_{11}$, the index $|U : I|$ must be 1, 11, 55, or 110 by inspecting the degrees of $G$ and the indices of maximal subgroups of $A_{11}$. Furthermore, $I/M \cong A_{11}, A_{10}, S_5,$ or $A_9$, respectively.

  If $\phi_i$ is an extension of $\theta$ for some $i$, Gallagher’s lemma and the Clifford theory show that $(\phi_i\tau)^G_i \in \text{Irr}(G')$ for all $\tau \in \text{Irr}(I/M)$. Therefore, $12|U : I|\tau(1) \in \text{cd}(G')$ for every $\tau \in \text{Irr}(I/M)$. A routine check shows that this is impossible. From now on we assume that no $\phi_i$ is an extension of $\theta$. In other words, $\phi_i(1) > 1$ for all $i$.

  First we assume $|U : I| = 1$ or 11. Then $I/M = A_{11}$ or $A_{10}$. Using Lemma 2.7, we have $\phi_i = \theta_i \tau_i$, where $\theta_i$ is a character of an irreducible projective representation of $I$ of degree 1 and $\tau_i$ is a character of an irreducible projective representation of
Let $I/M$. The fact that $\theta_i$ is not an extension of $\theta$ implies that $\tau_i$ is a character of a properly irreducible projective representation of $I/M$. As $12 | U : I|\phi_i(1) = 12 | U : I|\tau_i(1)$ divides some degree of $G$, we then see that $\tau_i(1) = 16$ by examining the degrees of properly projective representations of $A_{10}$ and $A_{11}$ in [2]. Therefore, we obtain $\phi_i M = 16 \theta_i$. By the Frobenius reciprocity, it follows that the multiplicity of each $\phi_i$ in $\theta^I$ is 16. Therefore,

$$\theta^I = \sum_i 16 \phi_i,$$

where $\phi_i$'s are distinct. So $\theta^I(1)$ is divisible by $16^2$, which is a contradiction since $\theta^I(1) = |I/M| = 10!/2$ or $11!/2$.

Finally, we assume $|U : I| = 55$ or 110. Then $\phi_i(1) = 2$ or 1 and $I/M = S_9$ or $A_9$, respectively. Since $\phi_i$ has degree 1 or 2, the quotient $I/Ker(\phi_i)$ has a faithful irreducible character of degree 1 or 2 and hence $I/Ker(\phi_i)$ does not have a composition factor isomorphic to $A_9$. Therefore, the same thing would happen to $I/\bigcap_i Ker(\phi_i)$. However,

$$\bigcap_i Ker(\phi_i) = Ker(\theta^I) = \bigcap_{x \in I}(Ker \theta)^x \leq M$$

and hence $I/M$ does not have a composition factor isomorphic to $A_9$, a contradiction.

**Case 2.** $|G' : U| = 66$ and $U/M \cong S_{10}$. Then $66 | U : I|\phi_i(1)$ divides some degree of $G$. Since $|U : I|$ is an index of a subgroup of $U/M \cong S_{10}$, either $|U : I| = 1$ or $|U : I| = 2$. Also, the quotient $I/M$ is $S_{10}$ or $A_{10}$.

If $\phi_i$ is linear for some $i$, then it is an extension of $\theta$ to $I$. Gallagher’s lemma then implies that $(\phi_i \tau)^G \in Irr(G')$ for all $\tau \in Irr(I/M)$. Since $I/M$ is $S_{10}$ or $A_{10}$, we can choose $\tau$ to be the irreducible character of degree 567. This will produce a forbidden degree of $G'$.

So we can assume $\phi_i(1) > 1$ for all $i$. Lemma 2.7 implies that $\phi_i = \theta_i \tau_i$, where $\theta_i$ is a character of an irreducible projective representation of $I$ of degree 1 and $\tau_i$ is a character of a properly irreducible projective representation of $I/M$, which is $S_{10}$ or $A_{10}$. Recall that $66 | U : I|\tau_i(1) = 66 | U : I|\phi_i(1)$ divides some degree of $G$. Inspecting the degrees of $G$, we deduce that $\tau_i(1) = \phi_i(1) = 32$. It follows that $U = I$ and hence $I/M \cong S_{10}$. Since $\phi_i M = 32 \theta$, the Frobenius reciprocity implies that the multiplicity of each $\phi_i$ in $\theta^I$ is 32. Therefore,

$$\theta^I = \sum_i 32 \phi_i,$$

where $\phi_i$'s are distinct. So $10! = \theta^I(1)$ is divisible by $32^2$, which is a contradiction.

**Case 3.** $|G' : U| = 220$ and $U/M \cong (A_9 \times 3) : 2$. Then $220 | U : I|\phi_i(1)$ divides some degree of $G$. Inspecting the list of degrees of $G$, we see that $|U : I|\phi_i(1)$ divides 6, 14, or 16. Since $|U : I|$ is an index of a subgroup of $U/M \cong (A_9 \times 3) : 2$, it follows that $|U : I|$ must divide 6. In particular, $I/M$ has a normal subgroup isomorphic to $A_9$. 
If \(|U : I| = 3\) or \(6\), then \(\phi_i(1)\) divides 1 or 2. As in Case 1, \(I/\bigcap_i \ker(\phi_i)\) does not have a composition factor isomorphic to \(A_9\). This is a contradiction since \(\bigcap_i \ker(\phi_i) \leq M\) and \(I/M\) has a normal subgroup isomorphic to \(A_9\).

Thus, \(|U : I| = 1\) or 2 and let \(J \triangleleft I\) such that \(J/M \cong A_9\). Assume

\[
\theta' = e_1\delta_1 + e_2\delta_2 + \cdots + e_s\delta_s,
\]

where \(\delta_i\)'s are distinct characters in \(\text{Irr}(J/\theta)\). If \(\delta_j(1) = 1\) for some \(j\), then \(\theta\) extends to \(\delta_j \in \text{Irr}(J/\theta)\) and so by Gallagher’s lemma, \(\delta_j \tau \in \text{Irr}(J/\theta)\) for every \(\tau \in \text{Irr}(J/M)\). As \(J \leq I\), we deduce that \(\delta_j(1)\tau(1) = \tau(1)\) divides 6, 14 or 16 for all \(\tau \in \text{Irr}(J/M)\). Since \(J/M \cong A_9\), we can choose \(\tau\) to be the irreducible character of degree 27 to get a contradiction.

Now we can assume \(e_i > 1\) for all \(i\). Lemma 2.7 implies that \(\delta_i = \theta_i\tau_i\), where \(\theta_i\) is a character of an irreducible projective representation of \(J\) of degree 1 and \(\tau_i\) is a character of a properly irreducible projective representation of \(J/M\), which is \(A_9\), with degree \(e_i\). As \(\delta_i(1)\) divides 6, 14 or 16, inspecting the list of irreducible projective representations of \(A_9\), we deduce that \(e_i\) divides 14 for all \(i\) and hence \(\delta_i(1) = \tau_i(1) = 8\) for all \(i\). Hence, \(\theta\) is \(J\)-invariant and for any \(\varphi \in \text{Irr}(J/\theta)\), we have \(\varphi(1) = 2^3\). Now Lemma 2.6 yields that \(J/M \cong A_9\) is solvable, which is impossible.

Case 4. \(|G' : U| = 462\) and \(U/M \cong (A_6 \times A_6) : 2^2\). We then have \(|U : I|\phi_i(1)\) divides 8 or 9 for every \(i\). Inspecting the list of maximal subgroups of \(A_6\), we deduce that \(I/M\) possesses a normal subgroup isomorphic to \(A_6 \times A_6\). Let \(J \triangleleft I\) such that \(J/M \cong A_6 \times A_6\). Then \(|U : I|\) divides 4. If \(|U : I| > 1\), then \(\phi_i(1)|8\) for all \(i\), and so by applying Lemma 2.6, we obtain a contradiction. Thus, \(I = U\). Consider the following subnormal series

\[
M \triangleleft K \triangleleft J \triangleleft I, \text{ where } K/M \cong A_6 \cong J/K.
\]

We deduce that \(\theta \in \text{Irr}(M)\) is \(K\)-invariant. As \(K\) is subnormal in \(I\), if \(\varphi \in \text{Irr}(K/\theta)\), then \(\varphi(1)\) divides \(\phi_i(1)\) for some \(i\) so that \(\varphi(1)\) also divides either 8 or 9.

If \(\theta\) extends to \(\theta_0 \in \text{Irr}(K)\), then Gallagher’s lemma implies that \(\tau\theta_0 \in \text{Irr}(K/\theta)\) for all \(\tau \in \text{Irr}(K/M)\), and so \(\tau(1) = \tau(1)\theta_0(1)\) divides 8 or 9, which is impossible if we choose \(\tau \in \text{Irr}(K/M)\) with \(\tau(1) = 5\). Hence, for any \(\lambda \in \text{Irr}(K/\theta)\), we obtain \(\lambda(1) > 1\) and \(\lambda(1)\) divides 8 or 9. Let \(\gamma \in \text{Irr}(K/\theta)\) such that \(p|\gamma(1)\), where \(p = 2\) or \(p = 3\). As \(J/K \cong A_6\) has no maximal subgroup whose index divides 8 or 9, we deduce that \(\gamma\) is \(J\)-invariant and hence as \(p|\gamma(1)\), we have \(\phi(1)|p^3\) for any \(\phi \in \text{Irr}(V/\gamma)\). Thus, \(\gamma\) is \(J\)-invariant and for every \(\phi \in \text{Irr}(J/\gamma)\), \(\phi(1)\) is a \(p\)-power for a fixed prime \(p\), so that by Lemma 2.6, the quotient \(J/K\) is solvable, which is a contradiction.

Case 5. \(|G' : U| = 495\) and \(U/M \cong (A_8 \times A_4) : 2\). Then 495 divides \(\phi_i(1)\) divides some degree of \(G\) and hence \(|U : I|\phi_i(1)\) divides 6 or 9. Since \(|U : I|\) is an index of a subgroup of \(U/M \cong (A_8 \times A_4) : 2\), the index \(|U : I|\) must be 1, 2, 3, or 6 and therefore \(I/M\) has a normal subgroup isomorphic to \(A_8\).

Since \(\phi_i(1)\) divides 6 or 9, the quotient \(I/\ker(\phi_i)\) has a faithful irreducible character of degree dividing 6 or 9. Hence, \(I/\ker(\phi_i)\) does not have any composition factor isomorphic to \(A_8\) and the same thing happens to \(I/\bigcap_i \ker(\phi_i)\). Now
using $\bigcap_i \ker(\phi_i) \leq M$, we deduce that $I/M$ does not have a composition factor isomorphic to $A_8$, a contradiction.

Case 6. $|G' : U| = 792$ and $U/M \cong (A_7 \times A_5) : 2$. Then $792 | U : I| \phi_i(1)$ divides some degree of $|G|$ and hence $|U : I| \phi_i(1)$ divides 3. Since $|U : I|$ is an index of a subgroup of $U/M \cong (A_7 \times A_5) : 2$, the index $|U : I|$ must be 1. In other words, we have $\phi_i(1) | 3$ and $I/M \cong (A_7 \times A_5) : 2$.

Let $J < I$ such that $J/M \cong A_7$. If $\phi_i J \in \text{Irr}(J)$ for some $i$, then $\phi_i \tau \in \text{Irr}(I)$ for all $\tau \in \text{Irr}(I/J) = \text{Irr}(S_6)$. Since $\phi_i \tau$ lies above $\theta$, it follows that $(\phi_i \tau)^G(1) = \phi_i(1) \tau(1) | G' : I | \in \text{cd}(G')$ for every $\tau \in \text{Irr}(S_6)$. Taking $\tau(1) = 4$, we get a forbidden degree of $G'$.

So we can assume that $\phi_i J$ is reducible for all $i$. Recall that $\phi_i(1) | 3$. Hence, $\phi_i J$ is a sum of linear characters of $J$. Therefore,

$$J' \leq \bigcap_i \ker(\phi_i) \leq M.$$  

This implies that $J/M$ is abelian, contradicting the fact $J/M \cong A_7$.

Case 7. $|G' : U| = 5775$ and $U/M \cong 2^6 : 3^3 : S_4$. Then $U = I$ and $\phi_i$ is linear for all $i$. This means that $\theta$ extends to $I$ and therefore $\phi_i \tau \in \text{Irr}(I)$ for every $\tau \in \text{Irr}(I/M)$. Since $I/M$ is nonabelian, there is a nonlinear $\tau \in \text{Irr}(I/M)$. The irreducible character $(\phi_i \tau)^G$ of $G'$ then has degree larger than 5775, which is a contradiction since 5775 is the largest degree of $G$.

- The alternating group $A_{13}$. Keeping the setup and notation at the beginning of this section and inspecting Tables 3 and 4, we come up with the following cases.

Case 1. $|G' : U| = 13$ and $U/M \cong A_{12}$.

Subcase 1(a): $t = 1$. Then $I/M \cong A_{12}$. If $e_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(I)$ so that by Lemma 2.2, $\tau \theta_0$ is a constituent of $\theta^I$ for every $\tau \in \text{Irr}(I/M)$. By choosing $\tau \in \text{Irr}(I/M)$ with $\tau(1) = 2^9 \cdot 11$, we have

$$|G' : U| \tau(1) \theta_0(1) = 2^9 \cdot 11 \cdot 13 \cdot \theta(1)$$

divides some degree of $A_{13}$, which is impossible. Thus, $e_i > 1$ for all $i$ and hence each $e_i$ is a degree of a proper projective irreducible representation of $A_{12}$ by Lemma 2.7 and $13 e_i$ must divide some degree of $A_{13}$. Inspecting the list of projective irreducible degrees of $A_{12}$ in [2], we obtain $2^5 | e_i$ for all $i$. But then $2^5 \cdot 13$ divides no degree of $A_{13}$.

Subcase 1(b): $t > 1$. Then $|U : I|$ is divisible by the index of some maximal subgroup $T/M$ of $A_{12}$, where $I \leq T \leq U$. Inspecting the list of maximal subgroups of $A_{12}$, one of the following cases holds:

(i) $T/M \cong A_{11}$. Then $|U : T| = 12$ and $|T : I| \phi_i(1)$ divides one of the following numbers:

$$2^2 \cdot 3 \cdot 11, \ 2 \cdot 5 \cdot 11, \ 7 \cdot 11, \ 2^2 \cdot 3 \cdot 5, \ 2 \cdot 5^2.$$  

Assume $T = I$. If $e_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(I)$ and then $13 \cdot 12 \cdot \tau(1)$ must divide some degree of $A_{13}$, where $\tau \in \text{Irr}(I/M)$. By choosing $\tau \in \text{Irr}(A_{11})$ with $\tau(1) = 2^4 \cdot 7 \cdot 11$, we obtain a contradiction. Thus, $e_i > 1$ for all $i$,
and hence all $e_i$ are degrees of proper projective irreducible representations of $A_{11}$, where $13 \cdot 12 \cdot e_i$ divides some degree of $A_{13}$. Using [2], we deduce that $e_i = 2^4$ for all $i$. It follows that $c_i^2 = 2^8$ divides the order of $A_{11}$, which is impossible.

Thus, $I \leq T$. Hence, $|T : I|$ is divisible by the index of some maximal subgroup $R/M$ of $A_{11}$, where $I \leq R \leq T$, and $13 \cdot 12 \cdot |T : R| \cdot |R : I|\phi_i(1)$ divides some degree of $A_{13}$. Then either $R/M \cong A_{10}$ or $R/M \cong S_9$.

Assume $R/M \cong A_{10}$. Then $|R : I|\phi_i(1)$ divides $2^2 \cdot 3$, $2 \cdot 5$ or $7$.

If $|R : I| > 1$, then as $10$ is the smallest index of a maximal subgroup of $A_{10}$ and any other index is at least $45$, we deduce that $|R : I| = 10$ and $I/M \cong A_9$, so that all $\phi_i(1) = 1$. By Lemma 2.2, we deduce that $I/M$ is abelian which is impossible. Thus, we conclude that $I = R$. Now if $e_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(I)$ and then $13 \cdot 12 \cdot 11 \cdot \tau(1)$ must divide some degree of $A_{13}$, where $\tau \in \text{Irr}(I/M) = \text{Irr}(A_{10})$. By choosing $\tau \in \text{Irr}(A_{10})$ with $\tau(1) = 2^7 \cdot 3$, we obtain a contradiction. Thus, $e_i > 1$ for all $i$, and hence all $e_i$ are degrees of proper projective irreducible representations of $A_{10}$, where $13 \cdot 12 \cdot 11 \cdot e_i$ divides some degree of $A_{13}$. However, by [2], there is no such $e_i$.

Assume $R/M \cong S_9$. Then $|R : I|\phi_i(1)$ divides $2$. Hence, $I/M$ possesses a normal subgroup which is isomorphic to $A_9$. In particular, $I/M$ is nonsolvable, $\theta$ is $I$-invariant and every irreducible constituent of $\theta^I$ has degree a power of 2. We then obtain a contradiction by Lemma 2.6.

(ii) $T/M \cong S_10$. Then $|T : I|\phi_i(1)$ divides one of the following numbers:

$$5^2, \ 2^2 \cdot 3, \ 2^2 \cdot 5, \ 3 \cdot 5, \ 2 \cdot 7.$$ 

Inspecting the list of maximal subgroups of $S_{10}$, we deduce that $|T : I|$ divides 2 and so $I/M$ has a normal subgroup $W/M \cong A_{10}$. Then $\theta$ is $W$-invariant and

$$\theta^W = f_1\mu_1 + \cdots + f_k\mu_k,$$

where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, $\mu_i(1)$ divides one of the numbers above for every $i$. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Gallagher’s lemma, $\tau(1)\theta_0(1)$ is a degree of an irreducible constituent of $\theta^W$ for any $\tau \in \text{Irr}(W/M)$, so that $\tau(1)$ must divide one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_{10})$ with $\tau(1) = 2^7 \cdot 3$, we obtain a contradiction. Thus, $f_i > 1$ is a degree of a proper projective irreducible representation of $A_{10}$ for every $i$. It follows from [2] that $f_i = 2^4$ for all $i$. This leads to a contradiction as $f_i^2 = 2^8$ does not divide the order of $A_{10}$.

(iii) $T/M \cong (A_9 \times 3) : 2$. Then $|T : I|\phi_i(1)$ divides $4$ or $6$. Inspecting the list of maximal subgroups of $A_9$, we deduce that $I/M$ has a normal subgroup $W/M \cong A_9$. We have $\theta$ is $W$-invariant and write $\theta^W = f_1\mu_1 + \cdots + f_k\mu_k$, where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, we deduce that $\mu_i(1)$ divides $4$ or $6$ for every $i$. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Lemma 2.2, $\tau(1)\theta_0(1)$ is the degree of an irreducible constituent of $\theta^W$ for every $\tau \in \text{Irr}(W/M)$. It follows that $\tau(1)$ divides one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_9)$ with $\tau(1) = 3^3$, we obtain a contradiction. Thus, $f_i > 1$ is a degree of a proper projective irreducible representation of $A_9$ for every $i$. It follows from [2] that $f_i = 8$ for all $i$. But then this is a contradiction again as $\mu_i(1) = 8\theta(1) > 6$. 

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(iv) $T/M \cong (A_6 \times A_6) : 2^2$ or $T/M \cong (A_8 \times A_4) : 2$. Then $|T : I|\phi_i(1)$ divides 2. It follows that $T = I$ or $|T : I| = 2$. In both cases, we obtain that $I \leq T$ and $T/I$ is cyclic so that $I/M$ is nonsolvable. We then obtain a contradiction by Lemma 2.6.

Case 2. $|G' : U| = 78$ and $U/M \cong S_{11}$. Then $|U : I|\phi_i(1)$ divides one of the following numbers:

$$5^2 \cdot 11, \ 2^3 \cdot 3 \cdot 11, \ 2^2 \cdot 5 \cdot 11, \ 3 \cdot 5 \cdot 11, \ 2 \cdot 7 \cdot 11, \ 2^3 \cdot 3 \cdot 5, \ 2^2 \cdot 5^2, \ 2 \cdot 3^3.$$

Assume $t \leq 2$. Then $I/M$ possesses a normal subgroup $W/M \cong A_{11}$. We have $\theta$ is $W$-invariant and write $\theta^W = f_1\mu_1 + \cdots + f_k\mu_k$, where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, we deduce that for every $i$, $\mu_i(1)$ divides one of the numbers above. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Gallagher’s lemma, $\tau(1)\theta_0(1)$ is a degree of an irreducible constituent of $\theta^W$ for any $\tau \in \text{Irr}(W/M)$, so that $\tau(1)$ must divide one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_{11})$ with $\tau(1) = 2^4 \cdot 7 \cdot 11$, we obtain a contradiction. Thus, $f_i > 1$ is a degree of a proper projective irreducible representation of $A_{11}$ for all $i$. It follows from [2] that $2^4 | f_i$ for all $i$. But then this is a contradiction as $2^8 | f_1^2$ does not divide the order of $A_{11}$.

Assume $t > 2$. We then have $I \leq T \leq U$ and $T/M$ is isomorphic to a maximal subgroup of $S_{16}$ with $|U : T| > 2$. Then one of the following cases holds:

(i) $T/M \cong S_{10}$. Then $|U : T| = 11$ and $|T : I|\phi_i(1)$ divides one of the following numbers:

$$5^2, \ 2^3 \cdot 3, \ 2^2 \cdot 5, \ 3 \cdot 5, \ 2 \cdot 7.$$

Assume $|T : I| > 2$. Inspecting the list of maximal subgroup of $S_{10}$, we deduce that $I/M \cong S_9$ and then $10|\phi_i(1)| 20$ so that $\phi_i(1) | 2$ for all $i$. Now Lemma 2.6 yields a contradiction.

Thus, $|T : I| \leq 2$ so that $I/M$ possesses a normal subgroup $W/M \cong A_{10}$. We have that $\theta$ is $W$-invariant and

$$\theta^W = f_1\mu_1 + \cdots + f_k\mu_k,$$

where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, $\mu_i(1)$ divides one of the numbers above for every $i$. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Lemma 2.2, $\tau(1)\theta_0(1)$ is the degree of an irreducible constituent of $\theta^W$ for any $\tau \in \text{Irr}(W/M)$, so that $\tau(1)$ must divide one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_{10})$ with $\tau(1) = 35$, we obtain a contradiction. Thus, $f_i > 1$ is a degree of a proper projective irreducible representation of $A_{10}$ for every $i$. However, $A_{10}$ has no such degree dividing any number above.

(ii) $T/M \cong S_9 \times 2$. Then $|U : T| = 55$ and $|T : I|\phi_i(1)$ divides 3, 4 or 5. Inspecting the list of maximal subgroup of $S_9$, we deduce that $I/M$ must possess a normal subgroup $W/M \cong A_9$. We have that $\theta$ is $W$-invariant and write $\theta^W = f_1\mu_1 + \cdots + f_k\mu_k$, where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, $\mu_i(1)$ divides one of the numbers above for every $i$. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Lemma 2.2, $\tau(1)\theta_0(1)$ is a degree of an irreducible constituent of $\theta^W$ for any $\tau \in \text{Irr}(W/M)$, so that $\tau(1)$ must divide one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_9)$ with $\tau(1) = 3^3$, we obtain a contradiction. Thus, $f_i > 1$ is a
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degree of a proper projective irreducible representation of \( A_9 \) and \( f_i \leq 5 \) for all \( i \). However, it follows from [2] that there is no such degree.

(iii) \( T/M \cong S_3 \times S_3 \). Then \( |U : T| = 165 \) and \( |T : I| \phi_i(1) = 1 \) for every \( i \). Hence, \( I = T \) and all \( \phi_i(1) = 1 \) so that \( I/M \) is abelian by Lemma 2.2, which is impossible.

Case 3. \( |G' : U| = 286 \) and \( U/M \cong (A_{10} \times 3) : 2 \). Then \( |U : I| \phi_i(1) \) divides one of the following numbers:

\[
3 \cdot 5^2, \ 2^3 \cdot 3^2, \ 2^2 \cdot 3 \cdot 5, \ 3^2 \cdot 5, \ 2 \cdot 3 \cdot 7, \ 2^3 \cdot 5, \ 2^2 \cdot 3^2, \ 2^2 \cdot 7.
\]

Let \( M \triangleleft W \triangleleft U \) such that \( W/M \cong A_{10} \).

Assume first that \( t \leq 6 \). Then \( I/M \) possesses a normal subgroup \( W/M \cong A_{10} \). Then \( \theta \) is \( W \)-invariant and \( \theta^W = f_1 \mu_1 + \cdots + f_k \mu_k \), where \( \mu_i \in \text{Irr}(W/\theta) \). As \( W \leq I \), we deduce that for every \( i \), \( \mu_i(1) \) divides one of the numbers above. If \( f_j = 1 \) for some \( j \), then \( \theta \) extends to \( \theta_0 \in \text{Irr}(W) \) and so by Lemma 2.2, \( \tau(1) \theta_0(1) \) is the degree of an irreducible constituent of \( \theta^W \) for every \( \tau \in \text{Irr}(W/M) \), so that \( \tau(1) \) must divide one of the numbers above. However, by choosing \( \tau \in \text{Irr}(A_{10}) \) with \( \tau(1) = 2^5 \cdot 5 \), we obtain a contradiction. Thus, \( f_i > 1 \) is a degree of a proper projective irreducible representation of \( A_{10} \) for every \( i \). It follows from [2] that \( 2^4 | f_i \) for all \( i \). But then this is a contradiction as \( 2^4 \mid f_i^2 \) does not divide the order of \( A_{10} \).

Hence, \( t > 6 \) and so \( W \nmid I \). As \( W \triangleleft U \), we have \( I \leq WI \leq U \), and hence

\[
t = |U : WI| \cdot |WI : I| = |U : WI| \cdot |W : W \cap I|,
\]

where \( M \leq W \cap I \leq W \). As \( W/M \cong A_{10} \), \( t \) is divisible by the index of a maximal subgroup \( R/M \) of \( A_{10} \), where \( W \cap I \leq R \leq W \). Then one of the following cases holds:

(i) \( R/M \cong A_9 \). Then \( |W : R| = 10 \) and \( |R : W \cap I| \phi_i(1) \) divides 4 or 6. Inspecting the list of maximal subgroups of \( A_9 \), we deduce that \( W \cap I = R \) and hence \( \theta \) is \( R \)-invariant where \( R/M \cong A_9 \). Assume

\[
\theta^R = g_1 \lambda_1 + \cdots + g_l \lambda_l, \text{ where } \lambda_j \in \text{Irr}(R/\theta).
\]

Since the characters \( \lambda_j \)'s are constituents of the restrictions of \( \phi_i \) to \( R \), we have that \( \lambda_j(1) \) divides 4 or 6 for every \( j \). If \( \lambda_j(1) = 1 \) for some \( j \), \( \theta \) would extend to \( \lambda_j \) and therefore \( \lambda_j \tau(1) \in \text{Irr}(R/\theta) \) for all \( \tau \in \text{Irr}(R/M) \). By taking the irreducible character \( \tau \) of \( A_9 \) of degree 8, we get a contradiction.

Thus, we can assume \( \lambda_j(1) > 1 \) for all \( j \). Then \( g_j > 1 \) is a degree of a proper projective irreducible representation of \( A_9 \) by Lemma 2.7. However, it follows from [2] that there is no such degree dividing 4 or 6.

(ii) \( R/M \cong S_3 \). Then \( |W : R| = 45 \) and \( |R : W \cap I| \phi_i(1) = 1 \) for all \( i \). Hence, \( R \leq I \). Furthermore, all irreducible characters of \( R \) lying above \( \theta \) are linear. On the other hand, by Gallagher’s lemma, \( \phi_i R \tau \in \text{Irr}(R/\theta) \) for every \( \tau \in \text{Irr}(R/M) \). This is a contradiction.

Case 4. \( |G' : U| = 715 \) and \( U/M \cong (A_9 \times A_4) : 2 \). Then \( |U : I| \phi_i(1) \) divides one of the following numbers:

\[
30, \ 24, \ 21, \ 18, \ 16.
\]
Let $M < W < U$ such that $W/M \cong A_9$.

Assume $W \not< I$. Then $|W : W \cap I|$ divides $t$ and is divisible by the index of some maximal subgroup $R/M$ of $A_9$, where

$$M \leq W \cap I \leq R \leq W.$$ 

It follows that $R/M \cong A_9$ and hence $9 | t$ so that $\phi_i(1) \mid 2$ for all $i$ and $W \cap I = R$. Thus, $I$ is nonsolvable and $\phi_i(1)$ is a 2-power for all $i$, now Lemma 2.6 yields a contradiction.

Thus, $W \leq I$ so that $\theta$ is $W$-invariant and $\theta^W = f_1 \mu_1 + \cdots + f_k \mu_k$, where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, we deduce that for every $i$, $\mu_i(1)$ divides one of the numbers above. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Gallagher’s lemma, $\tau(1)\theta_0(1)$ is the degree of an irreducible constituent of $\theta^W$ for every $\tau \in \text{Irr}(W/M)$, so that $\tau(1)$ must divide one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_9)$ with $\tau(1) = 3^3$, we obtain a contradiction. Thus, $f_i > 1$ is the degree of a proper projective irreducible representation of $A_9$ for every $i$. It follows from [2] that $f_i = 2^3$ for all $i$. We then have $\mu_i(1) = f_i \theta(1)$ so that $\mu_i(1)/\theta(1) = 2^3$ for all $i$. Now Lemma 2.6 yields a contradiction again.

Case 5. $|G'| : U| = 1287$ and $U/M \cong (A_8 \times A_3) : 2$. Then $|U : I|\phi_i(1)$ divides one of the following numbers:

$$16, \ 10, \ 9, \ 7.$$ 

Let $M < W < U$ such that $W/M \cong A_9$.

Assume $W \not< I$. Then $|W : W \cap I|$ divides $t$ and is divisible by the index of some maximal subgroup $R/M$ of $A_9$, where $M \leq W \cap I \leq R \leq W$. It follows that $R/M \cong A_9$ and hence $9 | t$ so that $\phi_i(1) \mid 2$ for all $i$ and $W \cap I = R$. Thus, $I$ is nonsolvable and $\phi_i(1)$ is a 2-power for every $i$, which yields a contradiction by Lemma 2.6.

Thus, $W \leq I$ so that $\theta$ is $W$-invariant and $\theta^W = f_1 \mu_1 + \cdots + f_k \mu_k$, where $\mu_i \in \text{Irr}(W|\theta)$. As $W \leq I$, we deduce that for every $i$, the degree $\mu_i(1)$ divides one of the numbers above. If $f_j = 1$ for some $j$, then $\theta$ extends to $\theta_0 \in \text{Irr}(W)$ and so by Lemma 2.2, $\tau(1)\theta_0(1)$ is a degree of an irreducible constituent of $\theta^W$ for every $\tau \in \text{Irr}(W/M)$, so that $\tau(1)$ must divide one of the numbers above. However, by choosing $\tau \in \text{Irr}(A_8)$ with $\tau(1) = 21,

$$10, \ 9, \ 7.$$ 

Let $M < W < U$ such that $W/M \cong A_9$ and assume

$$\theta^I = e_1 \phi_1 + \cdots + e_l \phi_l$$

for some $l$.

Assume $W \not< I$. Then $|W : W \cap I|$ divides $t$ and is divisible by the index of some maximal subgroup $R/M$ of $A_9$, where $M \leq W \cap I \leq R \leq W$. It follows that

$$12, \ 10, \ 7.$$
\[ R/M \cong A_6 \] and hence \( 7 \mid t \) so that \( \phi_i(1) = 1 \) for all \( i \) and \( W \cap I = R \). Thus, \( I/M \) is nonsolvable and \( \phi_i(1) = 1 \) for all \( i \), and now Gallagher’s lemma will provide a contradiction. Thus, \( W \leq I \).

If \( \phi_j(1) = 1 \) for some \( j \), then \( \theta \) extends to \( \phi_j \). In particular, \( \theta \) extends to \( \phi_{1W} \).

Using Gallagher’s lemma, we obtain \( \phi_{1W} \tau \in \text{Irr}(W/\theta) \) for every \( \tau \in \text{Irr}(W/M) \). As \( W/M \cong A_7 \), this leads to a contradiction by taking \( \tau \) of degree 21. We have shown that \( W \leq I \) and \( e_j = \phi_j(1) > 1 \) for all \( j \in \{1, 2, \ldots, l\} \).

Assume \( \theta^W = f_1 \mu_1 + \cdots + f_l \mu_l \), where \( \mu_i \in \text{Irr}(W/\theta) \). As \( W \not\leq I \), we deduce that for every \( i \), \( \mu_i(1) \) divides one of the numbers above. If \( f_j = 1 \) for some \( j \), then \( \theta \) extends to \( \theta_0 \in \text{Irr}(W) \) and so by Lemma 2.2, \( \tau(1)\theta_0(1) \) is a degree of an irreducible constituent of \( \theta^W \) for any \( \tau \in \text{Irr}(W/M) \), so that \( \tau(1) \) must divide one of the numbers above. However, by choosing \( \tau \in \text{Irr}(A_7) \) with \( \tau(1) = 21 \), we obtain a contradiction again. Thus, for every \( i \), \( f_i > 1 \) is a degree of a proper projective irreducible representation of \( A_7 \). It follows from [2] that \( f_i = 4 \) or 6.

If \( \mu_1 \) extends to some \( \phi_j \in \text{Irr}(I/\theta) \), then \( \phi_j \tau \in \text{Irr}(I/\theta) \) for every \( \tau \in \text{Irr}(I/W) \). As \( I/W \cong S_6 \) and \( \phi_{iW} \) has degree dividing 12, 10, or 7, we get a contradiction by taking \( \tau \) of degree 8. Thus, \( \mu_1 \) does not extend to any \( \phi_j \). Consider a constituent \( \phi_i \) of \( \mu_1^I \). Let \( a > 1 \) be the multiplicity of \( \mu_1 \) in \( \phi_{iW} \). Then we get \( a \mu_1(1) = \phi_i(1) \).

As \( \phi_i(1) \) divides 12, 10, or 7 and \( \mu_1(1) = 4 \) or 6, we must have \( a \mu_1(1) = \phi_i(1) = 12 \).

In particular, \( \phi_{iW} = a \mu_1 \) and hence \( \mu_1 \) is invariant under \( I \). Furthermore, \( a = \phi_i(1)/\mu_1(1) = 12/\mu_i(1) \), which is either 2 or 3. This leads to a contradiction by Lemma 2.6.

6 Step 4: \( G = G' \times C_G(G') \)

First, we prove that \( M = 1 \) and therefore \( G' \cong H \) by Step 2. By Step 3 and Lemma 2.8, the index \( |M : M'| \) divides \( |\text{Mult}(A_{12})| = 2 \) and hence \( |M : M'| = 1 \) or 2. If \( |M : M'| = 2 \), then \( G'/M' \) is isomorphic to the universal cover of \( H \). When \( H = A_{12} \), this universal cover is \( S_{12} \) and it has an irreducible character of degree 7776. However, this is a contradiction since 7776 does not divide any member of \( \text{cd}(A_{12}) \). When \( H = A_{13} \), we obtain a similar contradiction as the degree 32 of \( S_{13} \) does not divide any member of \( \text{cd}(A_{13}) \).

We have proved that \( M = M' \). If \( M \) is abelian, then \( M = 1 \), as desired. So let us assume that \( M \) is not abelian. Let \( M/N = T_1 \times \cdots \times T_k \) be a chief factor of \( G' \), where each \( T_i \cong T \), a nonabelian simple group. As the \( T_i \)’s are permuted by \( G' \), we have that \( k \) is the index of a subgroup of \( G'/M \) (\( \cong A_{12} \) or \( A_{13} \)). If \( k > 1 \), then \( k \) is at least 12 and this is impossible since there is no degree of \( G \) divisible by \( p^{12} \) for some prime \( p \). So \( k \) must be 1.

As \( M/N \trianglelefteq G'/N \), the quotient \( (G'/N)/C_{G'/N}(M/N) \) is embedded in \( \text{Aut}(M/N) \). Therefore,
\[
\frac{G'/N}{M/N \times C_{G'/N}(M/N)} \leq \text{Out}(M/N),
\]
which is solvable by Schreier’s conjecture. The conclusion of Step 1 that \( G' = G'' \) then implies that
\[
G'/N = M/N \times C_{G'/N}(M/N) \cong H \times T,
\]
which is a contradiction. We conclude that $G' \cong H$.

Now we prove that $G = G' \times C_G(G')$. Assume by contradiction that $G' \times C_G(G')$ is a proper subgroup of $G$. Then $G$ induces on $G'$ some outer automorphism. This is impossible since $A_{12}$ has two irreducible characters of degree 1050 which are fused into $S_{12}$ but $2100 \notin \text{cd}(A_{12})$ and $A_{13}$ has two irreducible characters of degree 462 which are fused into $S_{13}$ but $924 \notin \text{cd}(A_{13})$. Thus, we must have $G = G' \times C_G(G')$. As $G' \cong H$ and $G/G'$ is abelian, the theorem is proved.

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