

- **Euler's Theorems** regarding degrees of vertices and whether or not a given graph has Euler circuits and paths
 1. **Euler circuits** (*closed unicursal tracings*)
 - (a) If a graph has *any* vertex of odd degree, then it cannot have an Euler circuit.
 - (b) If a graph is connected and every vertex has even degree, then it has at least one Euler circuit.
 2. **Euler paths** (*open unicursal tracings*)
 - (a) If a graph has *more than two* vertices of odd degree, then it cannot have an Euler path.
 - (b) If a graph is connected and has *exactly two* vertices of odd degree, then it has at least one Euler path. Any such path must start at one odd-degree vertex and end at the other one.
 3. **Degree in general**
 - (a) The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore must be an even number).
 - (b) The number of vertices of odd degree must be even.
- **Fleury's Algorithm** for finding an Euler circuit
 1. Make sure that the graph is connected and all the vertices have even degree.
 2. Start at any vertex.
 3. Travel through an edge if (a) it is not a bridge for the untraveled part of the graphs or (b) if there is no alternative.
 4. Label the edges in the order in which you travel them. When you can't travel any more, stop. (You are done!)
- How can one use Fleury's algorithm to find an Euler *path*?
- **Eulerizing a graph** (modifying a graph that contains no Euler circuit so that it contains an Euler circuit): Find all the vertices of odd degree. Add extra edges in such a way that the vertices of odd degree become vertices of even degree. The edges that we add must be duplicates of edges that already exist.

- **Semi-eulerizing a graph** (modifying a graph that contains no Euler path so that it contains an Euler path): Find all the vertices of odd degree. Add extra edges in such a way that all the vertices of odd degree *except two* become vertices of even degree. The edges that we add must be duplicates of edges that already exist.
- An **optimal** (semi-)eulerization includes as few duplicate edges as possible. If the (semi-)eulerization cannot be done by duplicating *only edges between vertices of odd degree*, then we will duplicate some edges between a vertex of odd degree and one with even degree.