- **Tools from §12.2 & 12.3**: Definition of convergence/divergences of series, telescoping series, the Divergence Test, the geometric series, the p-test
- **The Integral Test**: Consider the series $\sum a_n$, where $a_n = f(n)$, and $f(x)$ is a continuous, positive, decreasing function on $[1,\infty)$. The series converges if *and only if* $\int_1^\infty f(x)dx$ converges.
- **An error estimate**: Consider the *convergent* series $\sum a_i$, where $a_n = f(n)$. Write the sum of the series S as the nth partial sum s_n plus a remainder R_n. If $f(x)$ is a continuous, positive, decreasing function for $x \geq n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

- **The Comparison Test**: Consider $\sum a_n$ and $\sum b_n$ with positive terms.
  1. If $a_n \leq b_n$ for all n, and $\sum b_n$ converges, then $\sum a_n$ also converges.
  2. If $a_n \geq b_n$ for all n, and $\sum b_n$ diverges, then $\sum a_n$ also diverges.
  3. **Limit version**: If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ is positive and finite, then either both series converge, or both diverge.

- **The Alternating-Series Test**: If the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ $(b_n > 0)$ satisfies
  1. $b_{n+1} \leq b_n$ (for *all* n sufficiently large, i.e. n greater than or equal to some particular index value n_0. That is, the sequence is eventually decreasing.)
  2. $\lim_{n\to\infty} b_n = 0$,

  then the series is convergent. If you write the sum of the series S as $s_N + R_N$, then $|R_N| \leq b_{N+1}$ if $N \geq n_0$.

- **The Ratio Test**
  (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* [1]

  (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (including $L = \infty$), the series $\sum_{n=1}^{\infty} a_n$ diverges.

  (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, no conclusion.

- **The Root Test**
  (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

  (ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ (including $L = \infty$), then the series $\sum_{n=1}^{\infty} a_n$ diverges.

  (iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, no conclusion.

[1] Absolute convergence means $\sum_{n=1}^{\infty} |a_n|$ converges (and $\sum_{n=1}^{\infty} a_n$ converges).