

- **Definition:** An **inner product** (on a vector space  $S$  over  $\mathbb{C}$ ) is a function that assigns to each ordered pair of elements  $a, b \in S$  a scalar  $\langle a, b \rangle \in \mathbb{C}$  such that for all  $a, b, c \in S$  and all scalars  $\alpha \in \mathbb{C}$  :

1.  $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$

2.  $\langle \alpha a, b \rangle = \alpha \langle a, b \rangle$

3.  $\langle b, a \rangle = \overline{\langle a, b \rangle}$

4.  $\langle a, a \rangle > 0$  if  $a \neq 0$

- **Definition:** A set of elements  $a_1, a_2, \dots, a_n \in S$  is **orthogonal** with respect to an inner product defined on  $S$  if

$$\langle a_i, a_j \rangle = 0 \text{ for } i \neq j.$$

In other words, a set of elements in a vector space  $S$  is *orthogonal* with respect to an inner product defined on  $S$  if the inner product of each pair of distinct elements is zero.

- **Definition:** A set of elements  $a_1, a_2, \dots, a_n \in S$  is **orthonormal** with respect to an inner product defined on  $S$  if

$$\langle a_i, a_j \rangle = 0 \text{ for } i \neq j, \text{ and } \langle a_i, a_j \rangle = 1 \text{ for } i = j.$$

In other words, a set of elements in a vector space  $S$  is *orthonormal* with respect to an inner product defined on  $S$  if the inner product of each pair of distinct elements is zero, and the inner product of each element with itself is one.