

**Course:** Advanced Engineering Math I

**Instructor:** Dr. Laura Gross

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1. Consider the equations

$$\begin{aligned}\frac{dA_1}{dt} &= -i\kappa_1 A_1 + i\bar{A}_1 A_2 - \epsilon\alpha A_1, \\ \frac{dA_2}{dt} &= -i\epsilon\kappa_2 A_2 + \frac{i}{2} A_1^2 - \epsilon\alpha A_2 + \epsilon\gamma_2,\end{aligned}$$

This is a model for complex amplitudes  $A_1$  and  $A_2$  of the electric fields for two frequencies of light in an optical cavity at (real) time  $t$ . The parameters  $\kappa_1$  and  $\epsilon\kappa_2$  are detuning parameters,  $\epsilon\gamma_2$  is a pumping term, and  $\epsilon\alpha A_1$  and  $\epsilon\alpha A_2$  are damping terms. All the parameter values are real. Write the system as four real equations in the unknowns  $(x_1(t), x_2(t), x_3(t), x_4(t)) = (\operatorname{Re}(A_1), \operatorname{Im}(A_1), \operatorname{Re}(A_2), \operatorname{Im}(A_2))$ .

2. Consider the equation  $ay''(x) + by'(x) + cy(x) = 0$ ,  $a$ ,  $b$ , and  $c$  constant.

- Write the general real-valued solution for the case in which  $b^2 - 4ac > 0$ .
- Write the ordinary differential equation as a system of equations in the variables  $u$  and  $v$ .
- Write the system you found in Problem (2b) as a matrix equation  $\mathbf{x}' = A\mathbf{x}$ , where

$$\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}. \text{ Identify the } 2 \times 2 \text{ matrix } A.$$

- Find the eigenvalues of your matrix  $A$ . Note they are the same as the roots of the characteristic equation in (2a).
  - Consider  $a = 1$ ,  $b = 1$ ,  $c = -6$ . Solve both the single second-order equation given and your matrix version, and show that the solutions are equivalent.
3. Systems of first-order equations can sometimes be transformed into a single equation of higher order. Consider the system  $x_1' = -2x_1 + x_2$ ,  $x_2' = x_1 - 2x_2$ .
- Solve the first equation for  $x_2$ , and substitute into the second equation, thereby obtaining a second-order equation for  $x_1$ . Solve this equation for  $x_1$ , and then determine  $x_2$  also.
  - Find the solution of the given system that also satisfies the initial conditions  $x_1(0) = 2$ ,  $x_2(0) = 3$ .
  - Solve the original system subject to the same boundary conditions by writing the system in matrix form, finding eigenvalues, etc. Show that the solution is equivalent to the one you found in (3b).

4. Find the solution to the initial-value problem  $\mathbf{x}' = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{x}$ ,  $\mathbf{x}(0) = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$ .