3450:438/538:001 **Homework 8** Fall 2007

Course: Advanced Engineering Math I
Instructor: Dr. Laura Gross
Recommended due date: Wednesday, October 17, 2007

THIS HOMEWORK IS NOT FOR COURSE CREDIT. However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

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1. Consider the linear homogeneous boundary-value problem (BVP) $y'' + y = 0$, $y(0) = 0$, $y(1) = 0$.
   (a) Solve the BVP.
   (b) What is the dimension of the vector space of solutions?

2. Consider the linear homogeneous boundary-value problem (BVP) $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$.
   (a) Solve the BVP.
   (b) Find a basis for the vector space of solutions.

3. Consider the linear homogeneous boundary-value problem (BVP) $y'' + \pi^2 y = 0$, $y(0) + y(1) = 0$, $y'(0) + y'(1) = 0$.
   (a) Solve the BVP.
   (b) Find a basis for the vector space of solutions.

4. Consider the equation
$$y'' + y = \sin x. \tag{1}$$
   (a) Use the method of variation of parameters (or the method of undetermined coefficients, if you prefer) to solve equation (1).
   (b) Solve equation (1), subject to the boundary conditions $y(0) = 0$, $y(\pi) = \pi/2$.
   (c) Solve equation (1), subject to the boundary conditions $y(0) = 0$, $y(\pi/2) = 1$.
   (d) Solve equation (1), subject to the boundary conditions $y(0) = 0$, $y(\pi) = 1$.
   (e) Which case from (4b)–(4d) is well-posed? Which is underdetermined? Which is overdetermined?

5. Let $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$, $L : \mathbb{R}^2 \to \mathbb{R}^3$.
   (a) Is L 1-to-1? Show your work.
   (b) Is L onto? Show your work.

(OVER)

(c) Is L invertible? Show your work.

6. Let $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $L : \mathbb{R}^3 \to \mathbb{R}^2$.

   (a) Is L 1-to-1? Show your work.

   (b) Is L onto? Show your work.

   (c) Is L invertible? Show your work.

7. Let $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} -i & 2+2i \\ -1+i & 3 \\ -3i & 1 \end{bmatrix}$.

   What is the matrix A^* corresponding to the adjoint operator of L with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1\bar{v}_1 + u_2\bar{v}_2$ on $\mathbb{C}^2$ and the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1\bar{v}_1 + u_2\bar{v}_2 + u_3\bar{v}_3$ on $\mathbb{C}^3$? (Note $L : \mathbb{C}^2 \to \mathbb{C}^3$, and $L^* : \mathbb{C}^3 \to \mathbb{C}^2$.) Prove the adjoint has the desired property
   $$\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle.$$

8. Consider the ODE $y''(x) = x^2 + A$ on $-a < x < a$, subject to boundary conditions $y'(-a) = 0$ and $y'(a) = 0$. A is a constant (scalar) parameter.

   (a) Write the ODE in operator form $L(y) = f$. Identify L and f.

   (b) Find the adjoint operator L^* and appropriate boundary conditions for the adjoint problem. By definition we need
   $$\langle L(y), w \rangle = \langle y, L^*(w) \rangle, \tag{2}$$
   where we use the inner product
   $$\langle u, v \rangle = \int_{-a}^{a} u(x)v(x)\, dx.$$
   Hint: Write the left-hand side of equation (2) in this problem in integral form, and integrate by parts twice to get the form on the right-hand side by choosing $L^*(w)$ and boundary conditions on w appropriately.

   (c) State the homogeneous adjoint boundary-value problem. Solve it to find $\mathcal{N}(L^*)$, subject to the boundary conditions.

   (d) Solve the *homogeneous version* of the *original* boundary-value problem, i.e. Find $\mathcal{N}(L)$, subject to the boundary conditions.

   (e) Explain which Fredholm Alternative holds for this problem.

   (f) If there is a solvability condition, find it without actually solving the problem. Then find the solution(s) to the original (nonhomogeneous) problem, if any.