

Course: Advanced Engineering Math I

Instructor: Dr. Laura Gross

Recommended due date: Wednesday, October 10, 2007

THIS HOMEWORK IS NOT FOR COURSE CREDIT. However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

1. Consider the vector space M_{22} of 2×2 matrices with real components. Let A and B be

$$\text{in } M_{22}: A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

(a) Prove that $\langle A, B \rangle$, defined as follows, is an inner product on M_{22} : $\langle A, B \rangle = ae + bf + cg + dh$.

(b) Calculate $\left\langle \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 9 & 0 \end{bmatrix} \right\rangle$ under this inner product.

2. Calculate the inner product $\langle (2 + 3i, -1 + 5i), (1 + i, -i) \rangle$ under the inner product on \mathbb{C}^2 defined in class.

3. Show that the real-valued functions $f(x) = 3x - 2$ and $g(x) = x$ are orthogonal in P_1 under the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$.

4. Let $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 4x_2y_2$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

(a) Prove that $\langle \mathbf{x}, \mathbf{y} \rangle$, defined as above, is an inner product on the vector space \mathbb{R}^2 .

(b) Prove that $B = \{(-8, 2), (1, 1)\}$ is an *orthogonal basis* of \mathbb{R}^2 with respect to inner product defined above. Prove B is not an *orthonormal set* in \mathbb{R}^2 .

(c) Write $(-6, 4)$ as a linear combination of the elements in the basis B of Problem (4b).

5. $\mathcal{C}[-L, L]$ is a vector space over \mathbb{R} , where L is a fixed constant. Let $f(x), g(x) \in \mathcal{C}[-L, L]$, and let

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^L f(x)g(x) dx. \quad (1)$$

(a) Prove that the function (1) defines an inner product.

(b) Let

$$S = \left\{ f(x) \mid f(x) = \frac{1}{\sqrt{2}} \text{ or } \sin\left(\frac{n\pi x}{L}\right) \text{ or } \cos\left(\frac{n\pi x}{L}\right), n = 1, 2, \dots \right\}.$$

Write down all the integral formulas that must hold if S is an *orthonormal set* in $\mathcal{C}[-L, L]$ with respect to the inner product (1).

* In fact, S in Problem (5b) is an (infinite) orthonormal *basis* for $\mathcal{C}[-L, L]$ (and, in fact, a wider class of functions) with respect to the inner product (1). As such, the integral identities from Problem (5b) *do* hold. Also as such, any $g(x) \in \mathcal{C}[-L, L]$ can be written as a linear combination of the elements of S :

$$g(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where

$$a_0 = \left\langle g(x), \frac{1}{\sqrt{2}} \right\rangle, \quad a_n = \left\langle g(x), \cos\left(\frac{n\pi x}{L}\right) \right\rangle, \quad b_n = \left\langle g(x), \sin\left(\frac{n\pi x}{L}\right) \right\rangle$$

as expected for an expansion in an orthonormal basis. We study these so-called Fourier series in AEM II. They are widely used in signal processing, wave propagation, heat transfer, vibrations, and many other applications.

6. Consider the system $-3x_1 + x_2 - x_3 + x_4 + x_5 = 0$, $x_2 + x_3 + 4x_5 = 0$, $-3x_3 + 2x_4 + x_5 = 0$.
- Perform Gaussian elimination to get $[A|\mathbf{b}]_R$ — the augmented matrix in RREF.
 - Find $\text{Rank}(A)$.
 - Write all solutions.
 - * (Optional) Using substitution, verify the solutions.
 - State a basis for $\mathcal{N}(A)$.
 - * (Optional) Prove it is a basis.
 - State the dimension of $\mathcal{N}(A)$.

7. Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \end{bmatrix}$.

- Find solvability condition(s) for $A\mathbf{x} = \mathbf{b}$, or state that there is none.
- Prove $\text{Col}(A)$ is a subspace of \mathbb{R}^3 . (Note this is true for *every* matrix A .)
- Find a basis for $\text{Col}(A)$. What is the dimension of $\text{Col}(A)$?
- Show the system is consistent for $\mathbf{b} = [-1 \ -23]^T$ in light of Problem (7a).
- Solve $A\mathbf{x} = \mathbf{b}$ for \mathbf{b} given in Part (7d).
- Show the solution in Part (7e) has the structure $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$.
 - * (Optional) Verify by substitution that $\mathbf{x} = \mathbf{x}_h$ satisfies $A\mathbf{x} = \mathbf{0}$; verify by substitution that $\mathbf{x} = \mathbf{x}_p$ satisfies $A\mathbf{x} = \mathbf{b}$.