3450:438/538:001 **Homework 15** Fall 2007

Course: Advanced Engineering Math I
Instructor: Dr. Laura Gross
Recommended due date: Friday, December 7, 2007

THIS HOMEWORK IS NOT FOR COURSE CREDIT. However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

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1. Show that Stokes's Theorem restricted to 2-D is Green's Theorem. Hint: Consider the vector field to have the form $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} + 0\mathbf{k}$, and let S be a surface in the xy-plane: a planar region D.

2. Evaluate $\int_{\mathcal{C}} x^2y^2\, dx + 4xy^3\, dy$ where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,3)$, and $(0,3)$ traversed counterclockwise.

3. Consider the part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$.

   (a) What is its mass if its density is $\rho(x,y,z) = y^2 + z^2$ g/cm^2?
   (b) What is its surface area?

4. A fluid with density 2 units of mass per volume flows with velocity
$$\mathbf{v} = -\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j} + \frac{3}{2}z\mathbf{k}.$$
   Find the rate of flow upward across the hemisphere $z = \sqrt{16 - x^2 - y^2}$.

5. Gauss' Law says the net charge Q enclosed by a closed surface S is equal to $\varepsilon_0 \int\int_S \mathbf{E} \cdot d\mathbf{S}$, where ε_0 is the (constant) permittivity of free space, and the surface integral is the electric flux through S. Find the net charge enclosed by the cube with vertices at $(\pm 1, \pm 1, \pm 1)$ if $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

6. Use Stokes's Theorem to evaluate $\int\int_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F} = x^2y^3z\mathbf{i} + \sin(xyz)\mathbf{j} + xyz\mathbf{k}$, and S is the part of the cone $y^2 = x^2 + z^2$ that lies between the planes $y = 0$ and $y = 3$ oriented in the direction of the positive y-axis.

7. Verify that Stokes' Theorem is true for $\mathbf{F} = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$, and $\mathcal{C}$ is the circle: $x^2 + y^2 = 16$, $z = 5$.

8. Verify that Divergence Theorem is true for $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$, and E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

9. Let E be a solid region, and let S be the boundary surface of E, given with positive (outward) orientation. Consider $\mathbf{F} = u\nabla v$, whose components have continuous first partial derivatives on an open region containing E. Show that $\iiint_E (\nabla u \cdot \nabla v + u\Delta v)\, dV = \iint_S u\nabla v \cdot dS$ (Green's First Identity).